Numerical methods for porous metals with deformation-induced anisotropy

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Abstract

A constitutive model for porous metals subjected to general three-dimensional finite deformations is presented. The model takes into account the evolution of porosity and the development of anisotropy due to changes in the shape and the orientation of the voids during deformation. Initially, the pores are assumed to be ellipsoids distributed randomly in an elastic–plastic matrix (metal). This includes also the special case in which the initial shape of the voids is spherical and the material is initially isotropic. Under finite plastic deformation, the voids are assumed to remain ellipsoids but to change their volume, shape and orientation. At every material point, a “representative” ellipsoid is considered and the homogenized continuum is assumed to be locally orthotropic, with the local axes of orthotropy coinciding with the principal axes of the representative local ellipsoid. The orientation of the principal axes is defined by the unit vectors $n^{(1)}$, $n^{(2)}$, $n^{(3)} = n^{(1)} \times n^{(2)}$ and the corresponding lengths are $2a_1$, $2a_2$ and $2a_3$. The basic “internal variables” characterizing the state of the microstructure at every point in the homogenized continuum are given by the local equivalent plastic strain $\varepsilon^{p}$ in the metal matrix, the local void volume fraction (or porosity) $f$, the two aspect ratios of the local representative ellipsoid ($w_1 = a_3/a_1$, $w_2 = a_3/a_2$) and the orientation of the principal axes of the ellipsoid ($n^{(1)}$, $n^{(2)}$, $n^{(3)}$).

A methodology for the numerical integration of the elastoplastic constitutive model is developed. The problems of uniaxial tension, simple shear, plastic flow localization and necking in plane strain tension, and ductile fracture initiation at the tip of a blunt crack are analyzed in detail; comparisons with the isotropic Gurson model are made.

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1. Introduction

Kailasam and Ponte Castañeda [32] proposed a general constitutive theory to model the effective behavior and microstructure evolution in composites and porous materials with “particulate” microstructures, undergoing general three-dimensional, finite deformations. The theory is based on a rigorous
homogenization technique [46] and builds on earlier work [31,33,34,48,49]. It is applicable to heterogeneous materials consisting of randomly oriented and distributed ellipsoidal inclusions (or pores), which, in the most general case, can change their size, shape and orientation, as a consequence of the applied deformation. In addition, the “shape” and “orientation” of the center-to-center statistical distribution functions of the inclusions [47] can also evolve with the deformation. The special case of porous metals was considered in some detail by Kailasam and Ponte Castañeda [31] and Kailasam et al. [34,35]. For these materials, it was found [34] that the “distribution” effects were small, and it was observed that the approximation of fixing the evolution of the shape and orientation of the distribution function equal to that of the voids themselves had a small overall effect in the predictions, while greatly simplifying the calculations involved (especially for situations where the distribution ellipsoid and inclusions are not aligned). In this work, use will be made of this simplifying assumption—keeping in mind that it could easily be relaxed at the expense of slightly heavier computation times. In addition, all the voids will be taken here to have initially the same shape and orientation, even if the general theory [32] can be used to treat the more general case of several families of aligned pores (with different shape and orientation for each family). In particular, the general theory would allow treatment of the case of randomly oriented voids, but this would result in considerable increase in the number of internal variables required, somewhat analogous to the situation for polycrystals, for which a version of the theory has also been developed recently [16].

Thus, the pores will be assumed in this work to be initially ellipsoidal (all with identical shapes and orientations) and distributed randomly (with the same shape and orientation for the distribution as for the voids themselves) in an elastic–plastic matrix (metal). Under finite plastic deformation, the voids remain ellipsoidal but change their volume, shape and orientation with the “local” macroscopic deformation. In this connection, it is emphasized that the size of the voids is assumed to be much smaller than the scale of variation of the macroscopic fields, in such a way that any “representative volume element” of the porous metal deforms uniformly with the local fields. It then makes sense to introduce, at each point in the homogenized continuum, a “representative” ellipsoid with principal axes defined by the unit vectors \( \mathbf{n}^{(1)}, \mathbf{n}^{(2)}, \mathbf{n}^{(3)} = \mathbf{n}^{(1)} \times \mathbf{n}^{(2)} \) and corresponding principal lengths \( a_1, a_2, a_3 \). The homogenized continuum is locally orthotropic, with the local axes of orthotropy coinciding with the principal axes of the representative ellipsoid. The basic “internal variables” characterizing the state of the microstructure at every point in the homogenized continuum are the equivalent plastic strain in the matrix \( (\varepsilon^p) \), the local void volume fraction or porosity \( (f) \), the two aspect ratios of the local representative ellipsoid \( (w_1 = a_3/a_1 \text{ and } w_2 = a_3/a_2) \), and the orientation of the principal axes of the ellipsoid \( (\mathbf{n}^{(1)}, \mathbf{n}^{(2)}, \mathbf{n}^{(3)}) \). The yield function of the homogenized continuum depends on the stress tensor and the aforementioned state variables: the “associated” plastic flow rule is derived from the yield function by using the “normality” rule. The plastic model is completed by the evolution equations of the state variables. A distinguishing feature of the model is that the rotation of the local axes of orthotropy during plastic flow is properly accounted for.

Several aspects of the aforementioned homogenization problem have been considered also by Gologanu et al. [17,18] and Gărâjeu et al. [12,13]. In particular, constitutive equations for axisymmetric loading of porous metals with a perfectly-plastic matrix containing axisymmetric ellipsoidal cavities have been developed by Gologanu et al. [17,18], who carried out “upper bound” calculations by using kinematically admissible velocity fields. The same problem for a viscoplastic matrix has been considered by Gărâjeu et al. [12,13], who derived expressions for the effective “dissipation potential” by using kinematically admissible velocity fields. It should be emphasized that a limitation of these models is that the voids were loaded axisymmetrically, so that their orientation was assumed to remain fixed.

In the present paper we focus on the computational issues associated with the use of the constitutive model of Kailasam and Ponte Castañeda [31,32] for porous metals in problems involving finite plastic deformation. The numerical implementation of the anisotropic elastic–plastic model in a finite element program and an algorithm for the numerical integration of the elastoplastic equations are presented. The problems of uniaxial tension, simple shear, plastic flow localization and necking in plane strain tension, and
ductile fracture initiation at the tip of a blunt crack are analyzed in detail. Comparisons with the isotropic Gurson model in which the voids are assumed to remain spherical during plastic flow are made.

Standard notation is used throughout. Boldface symbols denote tensors the orders of which are indicated by the context. All tensor components are written with respect to a fixed Cartesian coordinate system with base vectors \( \mathbf{e}_i \), \( (i = 1, 2, 3) \); and the summation convention is used for repeated Latin indices, unless otherwise indicated. The prefixes \( \mathrm{tr} \) and \( \mathrm{det} \) indicate the trace and the determinant, respectively, a superscript ‘\( \mathbf{T} \)’ the transpose, a superposed dot the material time derivative, and the subscripts ‘s’ and ‘a’ the symmetric and anti-symmetric parts of a second-order tensor. Let \( \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \) be vectors, \( \mathbf{A}, \mathbf{B} \) second-order tensors, and \( \mathbf{C}, \mathbf{D} \) fourth-order tensors; the following products are used in the text: \( \mathbf{a} \cdot \mathbf{b} = a_i b_i \), \( (\mathbf{ab})_{ij} = a_i b_j \), \( (\mathbf{abcd})_{ijkl} = a_i b_j c_k d_l \), \( (\mathbf{A} \cdot \mathbf{a})_i = A_{ik} a_k \), \( (\mathbf{a} \cdot \mathbf{A})_i = a_k A_{ki} \), \( \mathbf{A} : \mathbf{B} = A_{ij} B_{ij} \), \( (\mathbf{A} \cdot \mathbf{B})_{ij} = A_{ik} B_{kj} \), \( (\mathbf{AB})_{ijkl} = A_{ij} B_{kl} \), \( \mathbf{a} \cdot \mathbf{A} \cdot \mathbf{b} = a_i A_{ij} b_j = (\mathbf{ab}) : \mathbf{A} \), \( (\mathbf{C} : \mathbf{A})_{ij} = C_{ijkl} A_{kl} \), \( (\mathbf{A} : \mathbf{C})_{ij} = A_{kl} C_{kl ij} \), \( \mathbf{A} : \mathbf{B} = A_{ij} C_{ijkl} B_{kl} \) and \( (\mathbf{C} : \mathbf{D})_{ijkl} = C_{ijpq} D_{pqkl} \). The inverse \( \mathbf{C}^{-1} \) of a fourth-order tensor \( \mathbf{C} \) that has the “minor” symmetries \( C_{ijkl} = C_{jikl} = C_{ijlk} \) is defined so that \( \mathbf{C} : \mathbf{C}^{-1} = \mathbf{C}^{-1} : \mathbf{C} = \mathbf{I} \), where \( \mathbf{I} \) is the symmetric fourth-order identity tensor with Cartesian components \( I_{ijkl} = (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})/2 \), \( \delta_{ij} \) being the Kronecker delta.

2. Description of the constitutive model

In this section, the anisotropic elastic–plastic constitutive model for porous metals is described. The voids are assumed to be initially ellipsoidal and uniformly distributed in the isotropic metal matrix; as a consequence the porous metal is initially locally orthotropic. When the porous material is subjected to finite plastic deformations, the voids are assumed to remain ellipsoidal, but to change their volume and shape. This includes also the special case in which the initial shape of the voids is spherical and the material is initially isotropic. At every point in the homogenized porous metal a “representative local ellipsoid” is defined. Let \( \mathbf{n}^{(1)}, \mathbf{n}^{(2)}, \mathbf{n}^{(3)} = \mathbf{n}^{(1)} \times \mathbf{n}^{(2)} \) be unit vectors in the directions of the principal axes of the local ellipsoid and \( (2a_1, 2a_2, 2a_3) \) the corresponding lengths of the principal axes. The aspect ratios of the local ellipsoid are defined as \( (w_1 = a_3/a_1, w_2 = a_3/a_2) \). For simplicity and because spatial distributions effects are not expected to be significant for porous materials, the assumption is made, within the context the estimates of Ponte Castañeda and Willis [47], that the “shape” and “orientation” of the two-point correlation function characterizing the distribution of the voids in space has the same shape and orientation as the voids themselves. Then, it can be assumed that, as the material deforms, both the voids and their distribution evolve with identical shapes and orientations. This allows the use of the simplified linear-elastic estimates of Willis [53,54], as was done by Ponte Castañeda and Zaidman [48] in their original treatment of microstructure evolution in porous metals. In particular, this means that the porous material develops and maintains locally orthotropic symmetry; the local axes of orthotropy are aligned with the axes of the local representative ellipsoid. The “internal variables” that characterize the local state of the homogenized porous metal are \( \mathbf{s} = \{ \mathbf{e}^p, f, w_1, w_2, \mathbf{n}^{(1)}, \mathbf{n}^{(2)}, \mathbf{n}^{(3)} \} \), where \( \mathbf{e}^p \) is the local equivalent plastic strain in the metal matrix, and \( f \) the local void-volume-fraction or porosity.

The elastic and plastic response of the porous materials are treated independently, and combined later to obtain the full elastic–plastic response. The rate-of-deformation tensor \( \mathbf{D} \) at every point in the homogenized porous material is written as

\[
\mathbf{D} = \mathbf{D}^e + \mathbf{D}^p,
\]

where \( \mathbf{D}^e \) and \( \mathbf{D}^p \) are the elastic and plastic parts.

The constitutive model is presented next in three parts. The first part 2.1 deals with the elastic response of the porous metal. The yield condition and the plastic flow rule are presented in the second part 2.2. The third part 2.3 is concerned with evolution laws for the internal variables. Finally, in part 2.4 the elastic and
plastic constitutive equations are combined in order to derive the rate form of the elastoplastic equations, which relate the rate of deformation \( \mathbf{D} \) to the Jaumman derivative \( \dot{\mathbf{\sigma}} \) of the Cauchy (true) stress tensor \( \mathbf{\sigma} \).

### 2.1. Elastic constitutive relations

A hypoelastic form is assumed for the elastic part of the rate-of-deformation tensor:

\[
\mathbf{D}' = \mathcal{M}^e : \dot{\mathbf{\sigma}},
\]

where \( \mathcal{M}^e \) is the effective elastic compliance tensor and \( \dot{\mathbf{\sigma}} \) is a rate of the Cauchy stress which is corotational with the spin of the voids, i.e.,

\[
\dot{\mathbf{\sigma}} = \dot{\mathbf{\omega}} \cdot \mathbf{\sigma} + \mathbf{\sigma} \cdot \dot{\mathbf{\omega}}
\]

\( \dot{\mathbf{\omega}} \) being the spin of the voids relative to a fixed laboratory frame, i.e., \( \dot{\mathbf{n}}^{(i)} = \mathbf{\omega} \cdot \mathbf{n}^{(i)}, \) \( i = 1, 2, 3 \). The antisymmetric tensor \( \mathbf{\omega} \), which corresponds to what is normally called the “microstructural spin”, is calculated in Section 2.3 on microstructure evolution (Eq. (20)).

Making use of the simplifying assumption discussed earlier about the shape and orientation of the void distribution, the effective compliance tensor may be written as [53]

\[
\mathcal{M}^e = \mathcal{M} + \frac{f}{1-f} \mathbf{Q}^{-1}.
\]

In this expression, \( \mathcal{M} \) is the elastic compliance tensor of the matrix material, which is the inverse of its elastic modulus tensor \( \mathbf{L} \):

\[
\mathbf{L} = 2\mu \mathbf{K} + 3\kappa \mathbf{J}, \quad \mathcal{M} = \mathbf{L}^{-1} = \frac{1}{2\mu} \mathbf{K} + \frac{1}{3\kappa} \mathbf{J} = \frac{1}{2\mu} \left( \mathbf{K} + \frac{1 - 2v}{1 + v} \mathbf{J} \right),
\]

where \( \mu \) and \( \kappa \) denote the elastic shear and bulk moduli of the matrix, \( v \) is the Poisson’s ratio of the matrix, \( \delta \) and \( \mathbf{I} \) the second- and symmetric fourth-order identity tensors, with Cartesian components \( \delta_{ij} \) (the Kronecker delta) and \( I_{ijkl} = (\delta_{ij}\delta_{kl} + \delta_{il}\delta_{kj})/2 \), \( f \) is the porosity, and \( \mathbf{S} \) is the well-known fourth-order Eshelby [10,11] tensor. The Eshelby tensor \( \mathbf{S} \) has the minor symmetries (i.e., \( S_{ijkl} = S_{jikl} = S_{ijlk} \)), whereas the microstructural fourth-order tensor \( \mathbf{Q} \) [53] has both the major (“diagonal”) \( Q_{ijkl} = Q_{klij} \) and minor symmetries of the elasticity tensor. It should be noted that \( \mathbf{S} \) depends on the Poisson’s ratio \( v \) of the matrix, the aspect ratios \( (w_1, w_2) \) and the orientation vectors \( (\mathbf{n}^{(1)}, \mathbf{n}^{(2)}, \mathbf{n}^{(3)}) \); also, \( \mathbf{Q} \) is proportional to the shear modulus \( \mu \) of the matrix and depends also on \( v, (w_1, w_2) \) and \( (\mathbf{n}^{(1)}, \mathbf{n}^{(2)}, \mathbf{n}^{(3)}) \). Expressions for the tensors \( \mathbf{S} \) and \( \mathbf{Q} \) are given in Appendix A.

It is important to emphasize that the components of \( \mathcal{M}^e \) in expression (4) are not constants; they depend on the porosity, and the shape and orientation of the voids, which evolve in time. It is also recalled that the hypoelastic form (2) is consistent, to leading order, with hyperelastic behavior, because the elastic strains in porous metals are small relative to the plastic strains ([1,40]).

### 2.2. Yield condition and plastic flow rule

The main ingredient in the derivation of the constitutive relations is the variational procedure of Ponte Castaña [46] which is used to estimate the effective properties of the nonlinear porous material in terms of an appropriate “linear comparison composite.” The properties of the relevant linear comparison composite are obtained from Hashin–Shtrikman estimates of [47] for composites with “particulate” microstructures.
In the original derivation [30,32], the elastic strains were neglected and ideal plasticity was considered as the appropriate limit of a nonlinearly viscous solid. The effective yield function can be written in the form [30,48]:

\[
\Phi(\sigma_s) = \frac{1}{1-f} \sigma : \sigma - \sigma_y^2(\varepsilon^p),
\]

where \(\sigma_y\) is the yield strength in tension of the matrix material. It should be noted that in the original derivation perfect plasticity (i.e., \(\sigma_y = \text{const.}\)) was assumed; here the metal matrix is assumed to harden isotropically and \(\sigma_y\) is taken to be a function of the equivalent plastic strain \(\varepsilon^p\) in the matrix material. In the above expression for the yield function, the fourth-order tensor \(\mathbf{m}\) corresponds to an appropriately normalized effective viscous compliance tensor \(\mathbf{m}\) for the fictitious linear comparison porous material and is defined as

\[
\mathbf{m} = m(f, w_1, w_2, \mathbf{n}^{(1)}, \mathbf{n}^{(2)}, \mathbf{n}^{(3)}) = 3|\mathcal{E}|_{v=1/2} = \frac{3}{2} \mathbf{K} + \frac{3}{1-f} |\mathbf{Q}^{-1}|_{v=1/2},
\]

where the expression for \(\mathcal{E}\) is precisely the same as in (4). However, because of the assumed plastic incompressibility of the matrix phase, the limit as \(v \to 1/2\) must be taken in the expression (4) for \(\mathcal{E}\). This limiting process, which is complicated by the fact that the hydrostatic component of \(\mathcal{E}\) blows up in the definition (5) of the microstructural tensor \(\mathbf{Q} = \mathcal{E} : (\mathbf{I} - \mathbf{S})\), can be evaluated more conveniently by considering the explicit expressions for \(\mathbf{Q}\) given in Appendix A. It follows that \(|\mathbf{Q}^{-1}|_{v=1/2}\) depends on \((w_1, w_2, \mathbf{n}^{(1)}, \mathbf{n}^{(2)}, \mathbf{n}^{(3)})\). In the most general case, \(\Phi\) exhibits orthotropic symmetry with symmetry axes aligned with the axes of the voids, i.e., aligned with the vectors \(\mathbf{n}^{(i)}\) \((i = 1, 2, 3)\). It is emphasized that the plastic behavior described by the macroscopic potential \(\Phi\) is fully compressible [48], in agreement with experimental observations.

The plastic rate-of-deformation tensor \(\mathbf{D}^p\) is obtained in terms of \(\Phi\) from the “normality” relation

\[
\mathbf{D}^p = \dot{\Lambda} \mathbf{N}, \quad \mathbf{N} = \frac{\partial \Phi}{\partial \sigma} = \frac{2}{1-f} \mathbf{m} : \sigma,
\]

where \(\dot{\Lambda} \geq 0\) is the plastic multiplier, which is determined from the “consistency condition” as discussed in Section 2.4.

In the special case where the voids are spherical (i.e., \(w_1 = w_2 = 1\)), the porous metal is macroscopically isotropic and the yield function takes the form

\[
\Phi(\sigma, \varepsilon^p) = \left(1 + \frac{2}{3} f \right) \left(\frac{\sigma_e}{1-f}\right)^2 + 9 f \left(\frac{p}{1-f}\right)^2 - \sigma_y^2(\varepsilon^p) = 0,
\]

where \(\sigma_e = \sqrt{3} \sigma^d : \sigma^d/2\) is the von Mises equivalent stress, \(\sigma^d = \sigma - p\delta\) is the stress deviator, and \(p = \sigma_{kk}/3\) is the hydrostatic stress. The form of the constitutive model for the special case \(w_1 = w_2 = 1\) is discussed in detail in Appendix B.

2.3. Evolution of the microstructure

When the porous material deforms plastically, the state variables evolve and, in turn, influence the response of the material. In the current application to porous metals, it is assumed that all the changes in the microstructure occur only due to the plastic deformation of the matrix, which changes the volume, the shape and the orientation of the voids. This is expected to be reasonable, because the elastic strains here are relatively small compared to the plastic strains. The evolution equations for these variables are then determined from the kinematics of the deformation by assuming that the evolution of the relevant internal variables is characterized by the average plastic deformation rate and spin fields in the void phase, which
are estimated consistently by making use of the homogenization procedure of Ponte Castañeda and Zaidman [48] and Kailasam and Ponte Castañeda [32].

### 2.3.1. Evolution of the equivalent plastic strain \(\varepsilon^p\) and porosity \(f\)

The evolution of \(\varepsilon^p\) is determined from the condition that the local “macroscopic” plastic work \(\sigma : D^p = \lambda \sigma : N\) equals the corresponding “microscopic” work \((1-f)\sigma_y \varepsilon^p\), i.e.

\[
\dot{\varepsilon}^p = \dot{\lambda} \frac{\sigma : N}{(1-f)\sigma_y} \equiv \dot{\lambda}g_1(\sigma, s).
\]

Since the presence of porosity in a metal can be viewed as some kind of “damage” in the material and any changes in porosity due to elastic deformations are small and fully recoverable, it is assumed that changes in \(f\) are due to volumetric plastic deformation rates \(D^p_{kk}\) only (as opposed to the total \(D_{kk}\)). In view of the plastic incompressibility of the matrix phase, the evolution equation for the porosity \(f\) follows easily from the continuity equation and is given by

\[
\dot{f} = (1-f)D^p_{kk} = \dot{\lambda}(1-f)N_{kk} \equiv \dot{\lambda}g_2(\sigma, s).
\]

### 2.3.2. Evolution of the local aspect ratios and the local axes of orthotropy

The aforementioned variational procedure of Ponte Castañeda [46] has been used to determine the average deformation rate and the average spin of the local ellipsoid in terms of the macroscopic plastic deformation rate \(D^p\) and the macroscopic continuum spin \(W\). In particular, Ponte Castañeda and Zaidman [48] and Kailasam and Ponte Castañeda [32] have shown that the average deformation rate \(D^v\) and the average spin \(W^v\) in the local representative ellipsoidal void are

\[
D^v = A : D^p \quad \text{and} \quad W^v = W - C : D^p,
\]

where \(A\) and \(C\) are the relevant fourth-order “concentration tensors” defined as

\[
A = \left[ I - (1-f)S_{[v_1=1/2]} \right]^{-1} \quad \text{and} \quad C = -(1-f)\Pi : A.
\]

Here \(\Pi\) is the fourth-order Eshelby [10,11] rotation tensor that determines the spin of an isolated void in an infinite linear viscous matrix and depends on the aspect ratios \((w_1, w_2)\) and the orientation vectors \((n^{(1)}, n^{(2)}, n^{(3)})\). The tensor \(\Pi\) is symmetric with respect to the first two indices and antisymmetric with respect to the last two, i.e., \(\Pi_{ijkl} = -\Pi_{jikl} = \Pi_{ijlk}\). An expression for the evaluation of \(\Pi\) is given in Appendix A.

It should be noted that the “concentration tensors” \(A\) and \(C\) have the same symmetries and antisymmetries as \(S\) and \(\Pi\) respectively, and both depend on \((f, w_1, w_2, n^{(1)}, n^{(2)}, n^{(3)})\). In the limit as \(f \to 0\) the expressions for \(A\) and \(C\) reduce to the corresponding formulae of Eshelby [10,11] for the case of an isolated void in an infinite incompressible matrix.

The evolution of the aspect ratios \((w_1, w_2)\) is determined as follows. Starting with the definition \(w_1 = a_3/a_1\), one finds

\[
\dot{w}_1 = w_1 \left( \frac{\dot{a}_3}{a_3} - \frac{\dot{a}_1}{a_1} \right) = w_1 (n^{(3)} : D^v \cdot n^{(3)} - n^{(1)} : D^v \cdot n^{(1)}) = w_1 (n^{(3)} n^{(3)} - n^{(1)} n^{(1)}) : D^v,
\]

where \(2a_i\) is the length of the \(i\)th principal axis of the local representative ellipsoid. Taking into account that \(D^v = A : D^p\) and \(D^p = \lambda N\), we can write the last equation as

\[
\dot{w}_1 = \dot{\lambda}w_1 (n^{(3)} n^{(3)} - n^{(1)} n^{(1)}) : A : N \equiv \dot{\lambda}g_3(\sigma, s).
\]
Similarly,
\[ \dot{w}_2 = \dot{\lambda}w_2 (n^{(3)}n^{(3)} - n^{(3)}n^{(2)}) : A : N \equiv \dot{\lambda}g_4(\sigma, s). \] (16)

We define next the evolution equations for the orientation vectors \( n^{(i)} \). Since the \( n^{(i)} \)'s are unit vectors, their time derivative can be written in the form
\[ \dot{n}^{(i)} = \omega \cdot n^{(i)}, \] (17)
where \( \omega \) is an antisymmetric tensor.

The local representative ellipsoid can be thought of as developing during plastic flow from a “reference spherical void” of radius \( a_0 \). The deformation gradient of the ellipsoidal void \( F^v(t) \) relative to the reference spherical void can be written as
\[ F^v(t) = F_0^v \cdot F_v(t), \] (18)
where \( F_0^v \) is the deformation gradient of the initial representative void relative to the reference spherical void, and \( F_v(t) \) the deformation gradient of the evolving ellipsoidal void relative to its initial shape. The special case where the voids are initially spherical corresponds to \( F_0^v = \delta \). The results that follow are valid for both the general case of initially ellipsoidal voids (\( F_0^v \neq \delta \)) and the special case of initially spherical voids (\( F_0^v = \delta \)).

Using Eq. (18), we can show easily that the corresponding average velocity gradient of the void \( L^v = D^v + W^v \) can be written as
\[ L^v = D^v + W^v = \dot{F}_v \cdot F_v^{-1} = \dot{F}_0^v \cdot F_0^{-1}. \] (19)

The orientation of the unit vectors \( n^{(i)} \) along the principal axes of the local representative ellipsoid coincide with the Eulerian axes of \( F^v \). Therefore, the spin \( \omega \) in (17) is determined by the well-known kinematical relationship (e.g. [5,24,44])
\[ \omega_{ij} = W_{ij}^v - \frac{\lambda_i^2 + \lambda_j^2}{\lambda_i^2 - \lambda_j^2}D_{ij}^v, \quad i \neq j, \quad \lambda_i \neq \lambda_j, \quad \text{(no sum over } i), \] (20)
where \( \lambda_i \) are the stretch ratios of \( F^v \), i.e.,
\[ \lambda_i = \frac{a_i}{a_0}, \quad i = 1, 2, 3 \quad \text{with } w_3 = 1. \] (21)

Therefore, Eq. (20) can be written as
\[ \omega_{ij} = W_{ij}^v + \frac{w_i^2 + w_j^2}{w_i^2 - w_j^2}D_{ij}^v, \quad i \neq j, \quad w_i \neq w_j, \quad \text{(no sum over } i). \] (22)

In relations (20) and (22), and for the rest of this section, primed quantities indicate components in a coordinate frame that coincides instantaneously with the principal axes of the local representative ellipsoid, as determined by the vectors \( n^{(i)} \) (e.g. \( D^p = D_{ij}^p n^{(i)} n^{(j)} \), etc.).

It is emphasized that Eq. (22) is valid for both initially spherical and initially ellipsoidal voids, and that while the evolution of the void will naturally depend on its initial shape and orientation (through the initial dependence of \( D^v \) and \( W^v \) on the instantaneuous initial shape and orientation), the spin \( \omega \) in (22) depends only on the current shape and orientation of the voids and is independent of the corresponding initial values.

The special case in which at least two of the aspect ratios are equal is discussed in detail later in this section.
Remark 1. Taking into account (12), we can write Eq. (19) as
\[ \hat{\mathbf{F}} : \mathbf{F}^{v-1} = \dot{\mathbf{A}} : (\mathbf{A} - \mathbf{C}) : \mathbf{N} + \mathbf{W}, \]  
which is the differential equation that together with the initial condition \( \mathbf{F}^v(0) = \mathbf{F}_0^v \) defines the deformation gradient of the local representative ellipsoid \( \mathbf{F}^v(t) \).

In finite element computations it is convenient to refer all tensor components with respect to a fixed Cartesian coordinate system. Therefore, it is useful to state Eq. (22) in “direct notation” (for \( i \neq j, w_i \neq w_j \))
\[ n^{(i)} : \omega : n^{(j)} = n^{(i)} : W^v : n^{(j)} + \frac{w_i^2 + w_i^2}{w_i^2 - w_j^2} n^{(i)} : \mathbf{D}^v : n^{(j)} \]  (no sum over \( i \)).  
(24)

Also, since \( \mathbf{D}^v \) is symmetric, we can write that
\[
\mathbf{n}^{(i)} : \mathbf{D}^v : \mathbf{n}^{(j)} = (\mathbf{n}^{(i)} : \mathbf{n}^{(j)}): \mathbf{D}^v = \frac{1}{2}(\mathbf{n}^{(i)} : \mathbf{n}^{(j)} + \mathbf{n}^{(j)} : \mathbf{n}^{(i)}) : \mathbf{D}^v. 
\]  
(25)

Therefore, we can write the microstructural spin \( \mathbf{\omega} = (\mathbf{n}^{(i)} : \mathbf{\omega} : \mathbf{n}^{(j)}) \mathbf{n}^{(i)} \mathbf{n}^{(j)} \) in direct notation as
\[ \mathbf{\omega} = \mathbf{W}^v + \frac{1}{2} \sum_{i, j=1 \atop i \neq j \atop w_i \neq w_j}^{3} \frac{w_i^2 + w_j^2}{w_i^2 - w_j^2} \left( (\mathbf{n}^{(i)} \mathbf{n}^{(j)} + \mathbf{n}^{(j)} \mathbf{n}^{(i)}) : \mathbf{D}^v \right) \mathbf{n}^{(i)} \mathbf{n}^{(j)} \]  
(26)

where the expression \( \mathbf{W} = (\mathbf{n}^{(i)} : \mathbf{W} : \mathbf{n}^{(j)}) \mathbf{n}^{(i)} \mathbf{n}^{(j)} \) as well as Eqs. (24) and (25) have been taken into account.

Taking into account that \( \mathbf{D}^v = \mathbf{A} : \mathbf{D}^p, \mathbf{W}^v = \mathbf{W} - \mathbf{C} : \mathbf{D}^p, \) and \( \mathbf{D}^p = \dot{\mathbf{A}} \mathbf{N} \) we can write the last equation as (with \( w_3 = 1 \))
\[ \mathbf{\omega} = \mathbf{W} - \frac{1}{2} \sum_{i, j=1 \atop i \neq j \atop w_i \neq w_j}^{3} \frac{w_i^2 + w_j^2}{w_i^2 - w_j^2} \left( (\mathbf{n}^{(i)} \mathbf{n}^{(j)} + \mathbf{n}^{(j)} \mathbf{n}^{(i)}) : \mathbf{A} : \mathbf{N} \right) \mathbf{n}^{(i)} \mathbf{n}^{(j)} \]  
(27)

Finally, it is convenient to introduce the so-called “plastic spin” \( \mathbf{W}^p \), which is defined as the spin of the continuum relative to the substructure, i.e., \( \mathbf{W}^p = \mathbf{W} - \mathbf{\omega} \). Using the last equation, we conclude that the plastic spin can be written as
\[ \mathbf{W}^p = \dot{\mathbf{A}} \mathbf{\Omega}^p, \]  
(28)

where
\[
\mathbf{\Omega}^p = \mathbf{C} : \mathbf{N} - \frac{1}{2} \sum_{i, j=1 \atop i \neq j \atop w_i \neq w_j}^{3} \frac{w_i^2 + w_j^2}{w_i^2 - w_j^2} \left( (\mathbf{n}^{(i)} \mathbf{n}^{(j)} + \mathbf{n}^{(j)} \mathbf{n}^{(i)}) : \mathbf{A} : \mathbf{N} \right) \mathbf{n}^{(i)} \mathbf{n}^{(j)}, \quad (w_3 = 1). 
\]  
(29)

In the case where the coordinate frame is aligned with the principal axes of the local representative ellipsoid, the components of last equation reduces to
\[
\Omega^p_{ij} = \left( C_{ijkl} - \frac{w_i^2 + w_j^2}{w_i^2 - w_j^2} A^i_{ijkl} \right) N^p_{kl}, \quad i \neq j, \ w_i \neq w_j, \ w_3 = 1 \quad \text{(no sum over} \ i, j). \]  
(30)
It should be noted that, when two of the aspect ratios are equal, say \( w_1 = w_2 \), the material becomes locally transversely isotropic about the \( n^{(3)} \)-direction, the components \( C_{123} \) vanish, and Eq. (29) leaves \( W_{12}^p \) indeterminate; since the value of \( W_{12}^p \) is inconsequential in this case, it can be set equal to zero (see [1]). Also, when all three aspect ratios are equal \( (w_1 = w_2 = w_3 = 1) \), the material is locally isotropic, the spin concentration tensor \( C \) vanishes, and Eqs. (28) and (29) imply that \( W^p = 0 \) [9].

**Remark 2.** For the special case of a two-dimensional problem in which the deformation is taking place in the \( x_1 - x_2 \) plane, \( \Omega^p \) is of the form

\[
\Omega^p = \omega^p ( -e_1 e_2 + e_2 e_1 ) = \omega^p ( -n^{(1)} n^{(2)} + n^{(2)} n^{(1)} ),
\]

(31)

where \( (e_1, e_2) \) are unit vectors along the \( x_1 \) and \( x_2 \) axes, and the quantity \( \omega^p = -e_1 \cdot \Omega^p \cdot e_2 = -n^{(1)} \cdot \Omega^p \cdot n^{(2)} \) according to (29) takes the value

\[
\omega^p = -e_1 \cdot (C : N) \cdot e_2 + \frac{1}{2} \frac{w_1^2 + w_2^2}{w_1^2 - w_2^2} (n^{(1)} n^{(2)} + n^{(2)} n^{(1)}) : A : N \quad \text{when} \quad w_1 \neq w_2
\]

and

\[
\omega^p = 0 \quad \text{when} \quad w_1 = w_2.
\]

(33)

It should be noted that the constitutive functions \( \Phi, N, g_1, g_2, g_3, g_4, \) and \( \Omega^p \) are isotropic functions of their arguments, i.e., they are such that

\[
\Phi(R \cdot \sigma \cdot R^T, f, w_1, w_2, R \cdot n^{(i)}) = \Phi(\sigma, f, w_1, w_2, n^{(i)}),
\]

(34)

\[
N(R \cdot \sigma \cdot R^T, f, w_1, w_2, R \cdot N(i)) = R \cdot n(\sigma, f, w_1, w_2, n^{(i)}) \cdot R^T
\]

(35)

for all proper orthogonal tensors \( R \). The mathematical isotropy of the aforementioned functions guarantees the invariance of the constitutive equations under superposed rigid body rotations. It should be emphasized, however, that the material is anisotropic, due to the tensorial character of the \( n^{(i)} \)'s.

It should be mentioned that Eq. (17) can be written also in the form

\[
\ddot{n}^{(i)} = 0,
\]

(36)

where \( \ddot{n}^{(i)} \) is the rate of \( n^{(i)} \) corotational with the voids, i.e., \( \ddot{n}^{(i)} = \dot{n}^{(i)} - \omega \cdot n^{(i)} \).

Taking into account that \( W = \omega + W^p \), we conclude that the Jaumann derivative \( \ddot{n}^{(i)} = \dot{n}^{(i)} - W \cdot n^{(i)} \) can be written as \( \dddot{n}^{(i)} = \ddot{n}^{(i)} - W^p \cdot n^{(i)} \). Therefore, in view of (36),

\[
\dddot{n}^{(i)} = -W^p \cdot n^{(i)} = -A \Omega^p \cdot n^{(i)}.
\]

(37)

It follows that the evolution equations for all the microstructural variables \( s \), as given by relations (10), (11), (15), (16) and (37), can be written compactly in the form

\[
\dddot{s} = A G(\sigma, s),
\]

(38)

where \( \dddot{s} = \{ \dddot{e}, \dddot{J}, \dddot{W}, \dddot{n}^{(1)}, \dddot{n}^{(2)}, \dddot{n}^{(3)} \} \) and \( G \) is a collection of suitable isotropic functions. The plastic multiplier \( A \) can be computed from the so-called “consistency condition” as described in the following section.

In summary, constitutive laws have now been developed to describe the behavior of the elastic–plastic porous material. In the elastic regime the behavior is characterized by Eqs. (2)–(5) and in the plastic regime by Eqs. (6)–(8). The evolution of the microstructural variables \( s \) is characterized by Eqs. (10), (11), (15), (16) and (37).
2.4. Rateform of the elastoplastic equations

The developed constitutive equations are now manipulated in order to derive an equation relating the Jaumann derivative of the stress tensor \( \dot{\mathbf{\sigma}} \) to the total deformation rate \( \dot{\mathbf{D}} \). The derivation is as follows.

Assuming plastic loading \( (\mathbf{A} > 0) \), substitution of \( \mathbf{D}^e = \mathbf{D} - \mathbf{D}^p = \mathbf{D} - \dot{\mathbf{A}} \mathbf{N} \) into (2) yields

\[
\dot{\mathbf{\sigma}} = \mathbf{L}^e : \mathbf{D} - \dot{\mathbf{A}} \mathbf{L}^e : \mathbf{N},
\]

where \( \mathbf{L}^e = \mathbf{M}^{-1} \). Since \( \Phi \) is an isotropic function, the “consistency condition” \( \dot{\Phi} = 0 \) can be written in the form [9] (see also Appendix C)

\[
\dot{\Phi} = \frac{\partial \Phi}{\partial \mathbf{\sigma}} : \dot{\mathbf{\sigma}} + \frac{\partial \Phi}{\partial \mathbf{s}} : \dot{\mathbf{s}} = 0,
\]

where \( \dot{\mathbf{s}} = (\dot{\mathbf{e}}^p, \dot{\mathbf{f}}, \dot{\mathbf{w}}_1, \dot{\mathbf{w}}_2, \mathbf{\dot{n}}(1), \mathbf{\dot{n}}(2), \mathbf{\dot{n}}(3)) \). In view of the fact that \( \mathbf{\dot{n}}(i) : \mathbf{N} = 0 \) (Eq. (36)), the last relation can be written as

\[
\mathbf{N} : \dot{\mathbf{\sigma}} - \dot{\mathbf{A}} \mathbf{H} = 0 \quad \text{or} \quad \dot{\mathbf{A}} = \frac{1}{\mathbf{H}} \mathbf{N} : \mathbf{\dot{\sigma}} \quad \text{(for} \; \mathbf{H} \neq 0),
\]

where

\[
\mathbf{H} = -\left( \frac{\partial \Phi}{\partial \mathbf{e}^p} \mathbf{g}_1 + \frac{\partial \Phi}{\partial \mathbf{f}} \mathbf{g}_2 + \frac{\partial \Phi}{\partial \mathbf{w}_1} \mathbf{g}_3 + \frac{\partial \Phi}{\partial \mathbf{w}_2} \mathbf{g}_4 \right).
\]

It should be noted that the sign of the “hardening modulus” \( \mathbf{H} \) determines whether the material is hardening or softening. In particular, \( \mathbf{H} > 0 \) implies that the material is instantaneously hardening (yield surface expands in stress space), whereas \( \mathbf{H} < 0 \) implies that the material is instantaneously softening (yield surface contracts in stress space); the limiting case \( \mathbf{H} = 0 \) corresponds to instantaneous “perfect plasticity” (neither hardening nor softening) (e.g., Lubliner [36]). Note that in the present model the dimensions of \( \mathbf{H} \) are “stress” raised to the third power.

An alternative expression for \( \dot{\mathbf{A}} \) is obtained if one substitutes \( \mathbf{\dot{\sigma}} \) from (39) in (42):

\[
\mathbf{N} : \mathbf{L}^e : \mathbf{D} - \dot{\mathbf{A}} \left( \mathbf{N} : \mathbf{L}^e : \mathbf{N} + \mathbf{H} \right) = 0 \quad \text{or} \quad \dot{\mathbf{A}} = \frac{1}{\mathbf{L}} \mathbf{N} : \mathbf{L}^e : \mathbf{D},
\]

where \( \mathbf{L} = \mathbf{H} + \mathbf{N} : \mathbf{L}^e : \mathbf{N} \).

**Remark 3.** Note that \( \mathbf{H} \) is of order (flow stress)\(^3\) and can be positive or negative; on the other hand, the term \( \mathbf{N} : \mathbf{L}^e : \mathbf{N} \) is positive, since \( \mathbf{L}^e \) is positive definite, and of order (elastic modulus)\(\times\) (flow stress)\(^2\). In metals, the elastic modulus is several orders of magnitude larger than the flow stress; therefore, the term \( \mathbf{N} : \mathbf{L}^e : \mathbf{N} \) dominates and \( \mathbf{L} = \mathbf{H} + \mathbf{N} : \mathbf{L}^e : \mathbf{N} \) is always positive.

For a stress state on the yield surface (i.e., such that \( \Phi(\mathbf{\sigma}, \mathbf{s}) = 0 \)), the requirement \( \dot{\mathbf{A}} > 0 \) defines the “plastic loading condition”

\[
\mathbf{N} : \mathbf{L}^e : \mathbf{D} > 0,
\]

whereas \( \mathbf{N} : \mathbf{L}^e : \mathbf{D} = 0 \) corresponds to “neutral loading” \( (\dot{\mathbf{A}} = 0) \), and \( \mathbf{N} : \mathbf{L}^e : \mathbf{D} < 0 \) to “elastic unloading” \( (\dot{\mathbf{A}} = 0 \text{ as well}) \).
Substitution of \( \dot{\Lambda} \) from (44) into (39) yields
\[
\ddot{\sigma} = \left( \mathcal{L}^e - \frac{1}{L} \mathcal{L}^e : \mathbf{N} \right) : \mathbf{D}.
\] 
(46)

The Jaumann derivative \( \bar{\sigma} \) is related to \( \dot{\sigma} \) by the following expression
\[
\bar{\sigma} = \dot{\sigma} + \sigma : \mathbf{W}^p - \mathbf{W}^p : \sigma = \dot{\sigma} + \dot{\Lambda} (\sigma \cdot \mathbf{\Omega}^p - \mathbf{\Omega}^p \cdot \sigma) = \dot{\sigma} + \frac{1}{L} (\sigma \cdot \mathbf{\Omega}^p - \mathbf{\Omega}^p \cdot \sigma)(\mathbf{N} : \mathcal{L}^e : \mathbf{D}).
\] 
(47)

Finally, substitution of \( \dot{\sigma} \) from (46) into (47) yields the desired equation
\[
\bar{\sigma} = \mathcal{L}^{ep} : \mathbf{D}, \quad \mathcal{L}^{ep} = \mathcal{L}^e - \frac{1}{L} (\mathcal{L}^e : \mathbf{N})(\mathcal{L}^e : \mathbf{N}) + \frac{1}{L} (\sigma \cdot \mathbf{\Omega}^p - \mathbf{\Omega}^p \cdot \sigma)(\mathcal{L}^e : \mathbf{N}),
\] 
(48)

provided that \( \mathbf{N} : \mathcal{L}^e : \mathbf{D} > 0 \) (plastic loading). It should be noted that \( \mathcal{L}^{ep} \) does not have the major ("diagonal") symmetry (i.e., \( \mathcal{L}^{ep}_{ijkl} \neq \mathcal{L}^{ep}_{klij} \)) in general because of the last term in the above expression, which is the contribution of the "plastic spin" to the tangent modulus.

When \( \mathbf{N} : \mathcal{L}^e : \mathbf{D} \leq 0 \) (neutral loading or elastic unloading), the corresponding equation is
\[
\bar{\sigma} = \mathcal{L}^e : \mathbf{D}.
\] 
(49)

3. Numerical implementation of the constitutive model

In this section, the numerical integration of the constitutive equations is described (see also Kailasam et al. [35]). In a finite element environment, the solution is developed incrementally and the constitutive equations are integrated at the element Gauss integration points. Let \( \mathbf{F} \) denote the deformation gradient tensor. At a given Gauss point, the solution \( \mathbf{F}, \sigma_n, s_n \) at time \( t_n \) as well as the deformation gradient \( \mathbf{F}_{n+1} \) at time \( t_{n+1} \) are known, and the problem is to determine \( \sigma_{n+1}, s_{n+1} \).

The time variation of the deformation gradient \( \mathbf{F} \) during the time increment \( [t_n, t_{n+1}] \) can be written as
\[
\mathbf{F}(t) = \Delta \mathbf{F}(t) \cdot \mathbf{F}_n = \mathbf{R}(t) \cdot \mathbf{U}(t) \cdot \mathbf{F}_n, \quad t_n \leq t \leq t_{n+1},
\] 
(50)

where \( \mathbf{R}(t) \) and \( \mathbf{U}(t) \) are the rotation and right stretch tensors associated with \( \Delta \mathbf{F}(t) \). The corresponding deformation rate \( \mathbf{D}(t) \) and spin \( \mathbf{W}(t) \) tensors are given by
\[
\mathbf{D}(t) \equiv [\dot{\mathbf{F}}(t) \cdot \mathbf{F}^{-1}(t)]_s = [\Delta \dot{\mathbf{F}}(t) \cdot \Delta \mathbf{F}^{-1}(t)]_s,
\] 
(51)

and
\[
\mathbf{W}(t) \equiv [\dot{\mathbf{F}}(t) \cdot \mathbf{F}^{-1}(t)]_a = [\Delta \dot{\mathbf{F}}(t) \cdot \Delta \mathbf{F}^{-1}(t)]_a,
\] 
(52)

where the subscripts ‘s’ and ‘a’ denote the symmetric and anti-symmetric parts, respectively, of a tensor.

If it is assumed that the Lagrangian triad associated with \( \Delta \mathbf{F}(t) \) (i.e., the eigenvectors of \( \mathbf{U}(t) \)) remains fixed in the time interval \( [t_n, t_{n+1}] \), it can be shown readily that
\[
\mathbf{D}(t) = \mathbf{R}(t) \cdot \dot{\mathbf{E}}(t) \cdot \mathbf{R}^T(t), \quad \mathbf{W}(t) = \mathbf{R}(t) \cdot \mathbf{R}^T(t)
\] 
(53)

and
\[
\bar{\sigma}(t) = \mathbf{R}(t) \cdot \dot{\sigma}(t) \cdot \mathbf{R}^T(t), \quad \bar{\mathbf{n}}(t) = \mathbf{R}(t) \cdot \dot{\mathbf{n}}(t),
\] 
(54)

where \( \mathbf{E}(t) = \ln \mathbf{U}(t) \) is the logarithmic strain relative to the configuration at \( t_n \), \( \dot{\sigma}(t) = \mathbf{R}^T(t) \cdot \mathbf{\sigma}(t) \cdot \mathbf{R}(t) \), and \( \dot{\mathbf{n}}(t) = \mathbf{R}^T(t) \cdot \mathbf{n}(t) \).
It is noted that at the start of the increment \((t = t_n)\)
\[
F_n = R_n = U_n = \delta, \quad \sigma_n = \sigma_n, \quad \hat{n}_n^{(i)} = n_n^{(i)}, \quad \text{and} \quad E_n = 0, \quad (55)
\]
whereas at the end of the increment \((t = t_{n+1})\)
\[
\Delta F_{n+1} = F_{n+1} - F_n = R_{n+1} \cdot U_{n+1} = \text{known}, \quad \text{and} \quad E_{n+1} = \ln U_{n+1} = \text{known}. \quad (56)
\]
Taking into account that \(\Phi, N, g_1, g_2, g_3, g_4\) and \(\Omega^p\) are isotropic functions of their arguments, the elastoplastic equations can be written in the form
\[
\hat{E} = \hat{E}^c + \hat{E}^p, \quad (57)
\]
\[
\hat{\sigma} = \hat{\sigma}^e : \hat{E}^c + \hat{A}[\hat{\sigma} \cdot \Omega^p(\hat{\sigma}, \hat{s}) - \Omega^p(\hat{\sigma}, \hat{s}) \cdot \hat{\sigma}], \quad (58)
\]
\[
\Phi(\hat{\sigma}, \hat{s}) = 0, \quad (59)
\]
\[
\hat{E}^p = \hat{A}N(\hat{\sigma}, \hat{s}), \quad (60)
\]
\[
\hat{\sigma} = \hat{\sigma}^e : \hat{E}^c + \hat{A}[\hat{\sigma} \cdot \Omega^p(\hat{\sigma}, \hat{s}) - \Omega^p(\hat{\sigma}, \hat{s}) \cdot \hat{\sigma}], \quad (58)
\]
where
\[
\hat{\sigma} = (\hat{\sigma}^e, f, w_1, w_2, \hat{n}_1^{(1)}, \hat{n}_1^{(2)}, \hat{n}_1^{(3)}), \quad \hat{E}^p = (f, w_1, w_2, \hat{n}_1^{(1)}, \hat{n}_1^{(2)}, \hat{n}_1^{(3)}). \quad (59)
\]
Remark 4. The corotational rates \(\hat{\sigma}^c\) and \(\hat{n}_1^{(i)}\) in the original equations \(\hat{\sigma} = \hat{\sigma}^e : \hat{E}^c + \hat{A}(\hat{\sigma} \cdot \Omega^p - \Omega^p \cdot \hat{\sigma})\) and \(\hat{n}_1^{(i)} = -\hat{A}\Omega^p \cdot \hat{n}_1^{(i)}\) are now replaced in (58) and (65) by the usual material time derivatives \(\hat{\sigma}^p\) and \(\hat{n}_1^{(i)}\). This is a consequence of the assumption that the Lagrangian triad associated with \(\Delta F(t)\) remains fixed in the time interval \([t_n, t_{n+1}]\) so that \(W = R \cdot R^T\), which implies in turn that \(\hat{\sigma} = R \cdot \hat{\sigma} \cdot R^T\) and \(\hat{n}_1^{(i)} = R \cdot \hat{n}_1^{(i)}\). It should be noted though that the aforementioned assumption on the Lagrangian triad is less “severe” than the usual assumption of “constant strain rate” over the time increment.

In a recent publication Kailasam et al. [35] used a forward Euler scheme in order to integrate numerically the above set of equations. This limits the magnitude of the strain increment that can be used to that of the yield strain, especially in problems where the principal directions of stress rotate substantially over an increment. As a consequence, extremely small increments had to be used in problems such as metal forming. In order to overcome this difficulty, an alternative approach is proposed in the following.

Eq. (57) and the evolution equation of porosity (62) can be integrated exactly:
\[
\Delta E = \Delta E^c + \Delta E^p, \quad \text{or} \quad \Delta E^c = \Delta E - \Delta E^p, \quad (66)
\]
and
\[
f_{n+1} = 1 - (1 - f_n) \exp(-\Delta E^p_{kk}), \quad (67)
\]
where the notation \(\Delta A = A_{n+1} - A_n\) is used, and \(\Delta E = E_{n+1} = \text{known}\).
A backward Euler scheme is used for the numerical integration of the flow rule (60):
\[
\Delta \mathbf{E}^p = \Delta A \mathbf{N}_{n+1}, \quad \mathbf{N}_{n+1} = N(\sigma_{n+1}, f_{a+1}, w_{2x}|_{n+1}, \hat{\mathbf{n}}^{(i)}_{n+1}). \tag{68}
\]

The evolution equation for \( \hat{\mathbf{n}}^{(i)} \) (65) is approximated by
\[
\dot{\hat{\mathbf{n}}}^{(i)} = -\Delta \mathbf{A} \mathbf{p}^r \cdot \mathbf{n}^{(i)} \quad \text{or} \quad \frac{d\hat{\mathbf{n}}^{(i)}}{dA} = -\mathbf{p}^r \cdot \mathbf{n}^{(i)}, \tag{69}
\]

which, in turn, can be integrated exactly to give
\[
\hat{\mathbf{n}}^{(i)}_{n+1}(A, A) = \exp(-\Delta A \mathbf{p}^r) \cdot \mathbf{n}^{(i)} \tag{70}
\]

**Remark 5.** Note that direct application of a forward Euler scheme in (65) would result in an expression of the form \( \hat{\mathbf{n}}^{(i)}_{n+1} = (\delta - \Delta A \mathbf{p}^r) \cdot \mathbf{n}^{(i)} \). The factor \( \delta - \Delta A \mathbf{p}^r \) is a two-term approximation of the orthogonal tensor \( \exp(-\Delta A \mathbf{p}^r) \) in (70). Use of this approach would not keep the \( \hat{\mathbf{n}}^{(i)} \)'s unit vectors.

**Remark 6.** Eq. (70) requires the evaluation of the exponential of the antisymmetric second order tensor \( -\Delta A \mathbf{p}^r \). The exponential of an antisymmetric second-order tensor \( \mathbf{A} \) \( (\mathbf{A}^T = -\mathbf{A}) \) is an orthogonal tensor that can be determined from the following formula, attributed to Gibbs [7]
\[
\exp(\mathbf{A}) = \delta + \sin \frac{a}{a} \mathbf{A} + \frac{1 - \cos a}{a^2} \mathbf{A}^2, \tag{71}
\]
where \( a = \sqrt{\mathbf{A} : \mathbf{A}/2} \) is the magnitude of the axial vector of \( \mathbf{A} \).

**Remark 7.** In the special case of a two-dimensional problem in which the motion is taking place in the \( x_1 - x_2 \) plane, the \( \hat{\mathbf{n}}^{(i)} \)'s can be written as
\[
\hat{\mathbf{n}}^{(1)} = \cos \theta \mathbf{e}_1 + \sin \theta \mathbf{e}_2, \quad \hat{\mathbf{n}}^{(2)} = -\sin \theta \mathbf{e}_1 + \cos \theta \mathbf{e}_2, \quad \hat{\mathbf{n}}^{(3)} = \hat{\mathbf{n}}^{(1)} \times \hat{\mathbf{n}}^{(2)}, \tag{72}
\]

where \( (\mathbf{e}_1, \mathbf{e}_2) \) are unit vectors along the \( x_1 \) and \( x_2 \)-axes. In this case, Eq. (70) is equivalent to
\[
\hat{\theta}_{n+1} = \hat{\theta}_{n} + \Delta \hat{\theta}, \quad \Delta \hat{\theta}(\Delta A) = -\Delta A \omega^p_n, \tag{73}
\]
where \( \omega^p_n \) is defined according to Eqs. (32) and (33).

Finally, a forward Euler method is used for the numerical integration of the elasticity equation (58) and the evolution equations of the equivalent plastic strain in the matrix \( \mathcal{E}^p \) (60) and the aspect ratios \( w_{2x} \):
\[
\dot{\sigma}_{n+1}(\Delta A, \Delta \mathbf{E}^p) = \dot{\sigma}^e - \mathcal{L}^e : \Delta \mathbf{E}^p + \Delta A(\sigma_n \cdot \mathbf{p}^r - \mathbf{p}^e \cdot \mathbf{n}^{(i)}), \tag{74}
\]
\[
\dot{\mathcal{E}}^p_{n+1}(\Delta \mathbf{E}^p) = \dot{\mathcal{E}}^p_n + \frac{\sigma_n \cdot \Delta \mathbf{E}^p}{(1-f_n)\sigma_n(\mathcal{E}^e_n)}, \tag{75}
\]
\[
w_{1+n+1}(\Delta \mathbf{E}^p) = w_{1+1} + w_{1+n}(\mathbf{n}^{(1)}_n \mathbf{n}^{(1)}_n - \mathbf{n}^{(1)}_n \mathbf{n}^{(1)}_n) : \mathbf{A}_n : \Delta \mathbf{E}^p, \tag{76}
\]
\[
w_{2+n+1}(\Delta \mathbf{E}^p) = w_{2+1} + w_{2+n}(\mathbf{n}^{(1)}_n \mathbf{n}^{(1)}_n - \mathbf{n}^{(2)}_n \mathbf{n}^{(2)}_n) : \mathbf{A}_n : \Delta \mathbf{E}^p, \tag{77}
\]

where \( \dot{\sigma}^e = \sigma_n + \mathcal{L}^e : \Delta \mathbf{E} = \text{known is the “elastic predictor”}, \) and use has been made of the fact that \( \sigma_n = \sigma_n, \mathbf{n}^{(i)} = \mathbf{n}^{(i)}_n \) and \( \mathcal{L}^e = \mathcal{L}^e_n \).

The integration algorithm can be summarized as follows. The quantities \( \Delta A \) and \( \Delta \mathbf{E}^p \) are chosen as the primary unknowns and the yield condition and the plastic flow rule.
\[
\Phi(\tilde{\sigma}_{n+1}(\Delta A, \Delta E^p), \hat{\sigma}_{n+1}(\Delta A, \Delta E^p)) = 0,
\]  
(78)

\[
\Delta E^p = \Delta A N_{n+1}(\Delta A, \Delta E^p),
\]  
(79)

are treated as the basic equations in which \(\tilde{\sigma}_{n+1}, \hat{\sigma}_{n+1}, \bar{\sigma}_{n+1}, \bar{w}_1, w_2, \bar{w}_3, f_n, \bar{f}_n\) and \(\bar{n}^{(i)}_{n+1}\) are defined by Eqs. (74)–(77) (67) and (70). Eqs. (78) and (79) are solved for \(\Delta A\) and \(\Delta E^p\) by using Newton’s method. Once \(\Delta A\) and \(\Delta E^p\) are found, Eqs. (74)–(77), (67) and (70) define \(\tilde{\sigma}_{n+1}, \hat{\sigma}_{n+1}, \bar{\sigma}_{n+1}, \bar{w}_1, w_2, \bar{w}_3, f_n, \bar{f}_n\) and \(\bar{n}^{(i)}_{n+1}\). Finally, \(\sigma_{n+1}\) and \(\bar{n}^{(i)}_{n+1}\) are computed from

\[
\sigma_{n+1} = R_{n+1} \cdot \bar{\sigma}_{n+1} \cdot R^T_{n+1}, \quad \text{and} \quad \bar{n}^{(i)}_{n+1} = R_{n+1} \cdot \bar{n}^{(i)}_{n+1},
\]  
(80)

which completes the integration process.

The matrix mapping of tensorial expressions such as those used in the paper are discussed in detail by Nadeau and Ferrari [38].

We conclude this section with a comparison of the present algorithm to the earlier approach of Kailasam et al. [35]. Eq. (74) above can be written as

\[
\tilde{\sigma}_{n+1} = \sigma^* - \Delta A \mathbf{\bar{\mathcal{L}}}_n^{\mathbf{\sigma}} : \dot{\mathbf{N}}_{n+1} + \Delta A(\sigma_n \cdot \mathbf{\Omega}_n^p - \mathbf{\Omega}_n^p \cdot \sigma_n).
\]  
(81)

The corresponding equation in the approach of Kailasam et al. [35] is

\[
\hat{\sigma}_{n+1} = \sigma^* - \Delta A \mathbf{\bar{\mathcal{L}}}_n^{\mathbf{\sigma}} : \mathbf{N}_n + \Delta A(\sigma_n \cdot \mathbf{\Omega}_n^p - \mathbf{\Omega}_n^p \cdot \sigma_n).
\]  
(82)

In (81) and (82), the first term on the right hand side is the elastic predictor \(\sigma^*\) and the last term is due to the rotation of the axes of orthotropy relative to the continuum; the second term \(\Delta A \mathbf{\bar{\mathcal{L}}}_n^{\mathbf{\sigma}} : \mathbf{N}\) is the “return” onto the yield surface. When there is little or no hardening and the principal directions of stress rotate substantially over an increment, it is possible that no \(\Delta A\) can be found in the forward Euler scheme (82), so that the yield condition \(\Phi(\tilde{\sigma}_{n+1}, \hat{\sigma}_{n+1}) = 0\) is satisfied, if the magnitude of \(\Delta E\) exceeds the yield strain (Ortiz and Popov [45]). In other words, a “return” in the direction of \(\mathbf{N}_n\) is not possible, because the “hyper-line” in the direction of \(\mathbf{N}_n\) that goes through \(\sigma^*\) in stress space never meets the yield surface. In the backward Euler scheme, however, a “return” onto the yield surface in the direction of \(\mathbf{N}_{n+1}\) is always possible, thus allowing for larger strain increments \(\Delta E\).

### 3.1. Geometrically nonlinear thin shell problems

The implementation of the proposed algorithm to problems of geometrically nonlinear thin shells is discussed in this section. Let \(\mathbf{n}\) be the local unit vector normal to the shell laminae. The deformation gradient \(\mathbf{F}_{n+1}\) is still kinematically defined at the end of a given time increment, and the quantities \(\mathbf{R}_{n+1}, \mathbf{U}_{n+1}\) and \(\Delta \mathbf{E} = \ln \mathbf{U}_{n+1}\) are determined as before. The orientation of the laminae at the end of the increment \(\mathbf{n}_{n+1}\) is known as well. In order to simplify the notation, we drop the subscript \(n+1\) from \(\mathbf{n}_{n+1}\), with the understanding that \(\mathbf{n} \equiv \mathbf{n}_{n+1}\) for the rest of this section. The zero normal stress condition \(\sigma_{n+1} \cdot \mathbf{n} = 0\) can be written also as

\[
\sigma_{n+1} \cdot \mathbf{n} = 0,
\]  
(83)

where \(\mathbf{n} = R^T_{n+1} \cdot \mathbf{n} = \mathbf{n} \cdot \mathbf{R}_{n+1} = \text{known}\). The normal component \(\Delta E^\text{nor} \equiv \mathbf{n} \cdot \Delta \mathbf{E} \cdot \mathbf{n}\) is now ignored, treated as an unknown, and determined in such a way that the zero normal stress condition (83) is satisfied. The strain increment \(\Delta \mathbf{E}\) can be written as

\[
\Delta \mathbf{E} = \Delta \mathbf{E}^\mathbf{E} + \Delta E^\text{nor} \mathbf{n} \mathbf{n},
\]  
(84)

where \(\Delta \mathbf{E} = \ln \mathbf{U}_{n+1} - (\mathbf{n} \cdot \ln \mathbf{U}_{n+1} \cdot \mathbf{n}) \mathbf{n} \mathbf{n}\) is treated as the known part of \(\Delta \mathbf{E}\), and \(\Delta E^\text{nor}\) is the unknown normal strain component.
The corresponding form of Eq. (74) is

\[ \hat{\sigma}_{n+1} = \hat{\sigma}^e - L_n^e : \Delta \mathbf{E}^p + \Delta \mathbf{A}(\sigma_n \cdot \mathbf{\Omega}_n^p - \mathbf{\Omega}_n^p \cdot \sigma_n) + \Delta E_{nor} L_n^e : (\hat{\mathbf{n}}), \]

(85)

where \( \hat{\sigma}^e = \sigma_n + L_n^e : \Delta \mathbf{E} \) is known as the “elastic predictor”.

The evolution equations of the state variables are written as before

\[ \hat{\sigma}_{n+1}^e - \hat{\mathbf{n}} \cdot (L_n^e : \Delta \mathbf{E}^p) \cdot \hat{\mathbf{n}} + \Delta \mathbf{A} \cdot (\sigma_n \cdot \mathbf{\Omega}_n^p - \mathbf{\Omega}_n^p \cdot \sigma_n) \cdot \hat{\mathbf{n}} + \Delta E_{nor} L_n^e : (\hat{\mathbf{n}}), \]

(86)

which implies that

\[ \Delta E_{nor}(\Delta \mathbf{A}, \Delta \mathbf{E}^p) = -\frac{1}{L_n^e} (\hat{\sigma}_{n+1}^e - \hat{\mathbf{n}} \cdot (L_n^e : \Delta \mathbf{E}^p) \cdot \hat{\mathbf{n}} + \Delta \mathbf{A} \cdot (\sigma_n \cdot \mathbf{\Omega}_n^p - \mathbf{\Omega}_n^p \cdot \sigma_n) \cdot \hat{\mathbf{n}}), \]

(87)

where \( \hat{\sigma}_{nor}^e = \hat{\mathbf{n}} \cdot \hat{\sigma}^e \cdot \hat{\mathbf{n}} \) and \( L_n^e = (\hat{\mathbf{n}}) : L_n^e \) are known quantities. Taking into account the last equation, we conclude that (85) defines \( \hat{\sigma}_{n+1} \) in terms of \( \Delta \mathbf{A} \) and \( \Delta \mathbf{E}^p \):

\[ \hat{\sigma}_{n+1}(\Delta \mathbf{A}, \Delta \mathbf{E}^p) = \hat{\sigma}^e - L_n^e : \Delta \mathbf{E}^p + \Delta \mathbf{A}(\sigma_n \cdot \mathbf{\Omega}_n^p - \mathbf{\Omega}_n^p \cdot \sigma_n) + \Delta E_{nor}(\Delta \mathbf{A}, \Delta \mathbf{E}^p) L_n^e : (\hat{\mathbf{n}}). \]

(88)

The evolution equations of the state variables are written as before

\[ \hat{e}_{p+1}^e(\Delta \mathbf{E}^p) = \hat{e}_{n}^e + \frac{\sigma_n : \Delta \mathbf{E}^p}{(1 - f_n)\sigma_n(\hat{e}_{n}^e)}, \]

(89)

\[ f_{n+1} = 1 - (1 - f_n) \exp(-\Delta \mathbf{E}_{ik}), \]

(90)

\[ w_1|_{n+1}(\Delta \mathbf{E}^p) = w_1|_n + w_1|_n(\mathbf{n}_n^{(1)}\mathbf{n}_n^{(1)} - \mathbf{n}_n^{(1)}\mathbf{n}_n^{(1)}) : \mathbf{A}_n : \Delta \mathbf{E}^p, \]

(91)

\[ w_2|_{n+1}(\Delta \mathbf{E}^p) = w_2|_n + w_2|_n(\mathbf{n}_n^{(2)}\mathbf{n}_n^{(2)} - \mathbf{n}_n^{(2)}\mathbf{n}_n^{(2)}) : \mathbf{A}_n : \Delta \mathbf{E}^p, \]

(92)

\[ \mathbf{\hat{n}}_n^{(i)}(\Delta \mathbf{A}) = \exp(-\Delta \mathbf{A}\mathbf{\Omega}_n^p) : \mathbf{n}_n^{(i)} \]

(93)

The integration is completed as before by treating the yield condition and the plastic flow rule as the basic equations for the determination of \( \Delta \mathbf{A} \) and \( \Delta \mathbf{E}^p \).

4. Applications

The constitutive model presented in the previous sections is implemented in the ABAQUS general purpose finite element program [20]. This code provides a general interface so that a particular constitutive model can be introduced as a “user subroutine” (UMAT). 1 The integration of the elastoplastic equations is carried out using the algorithm presented in Section 3. The finite element formulation is based on the weak form of the momentum balance, the solution is carried out incrementally, and the discretized nonlinear equations are solved using Newton’s method. In the calculations, the Jacobian of the global Newton scheme is approximated by the tangent stiffness matrix derived using the moduli \( L_n^e \) given by Eq. (48). Such an approximation of the Jacobian is first-order accurate as the size of the increment \( \Delta \rightarrow 0 \); it should be emphasized, however, that the aforementioned approximation influences only the rate of convergence of the Newton loop and not the accuracy of the results.

1 Copies of the computer code (UMAT) will be supplied upon request. Please address inquiries to Prof. Aravas at the e-mail address aravas@uth.gr.
The matrix material, with Young's modulus $E$ and Poisson's ratio $\nu$, exhibits isotropic hardening with

$$\sigma_y(\overline{e}^p) = \sigma_0 \left(1 + \frac{\overline{e}^p}{\epsilon_0}\right)^{1/n},$$

(94)

where $\sigma_0$ is the yield stress in tension, $n \geq 1$ is the hardening exponent, and $\epsilon_0 = \sigma_0/E$. The values $E = 300\sigma_0$ and $\nu = 0.3$ are used in the calculations. The voids are assumed to be initially spherical and uniformly distributed in the isotropic metal matrix with an initial porosity of $f_0 = 0.04$.

In this case the material is initially isotropic and the aspect ratios take the values $w_1 = w_2 = 1$ initially. In order to avoid numerical singularities at the beginning of the calculations, the following technique is used: when a material point deforms plastically for the first time, the average strain increment $\Delta \epsilon^p$ in the representative ellipsoid is determined by using the deformation-rate–concentration tensor $A$ as $\Delta \epsilon^p = A : \Delta \epsilon^p$, the unit vectors $\mathbf{n}_i$ are then identified with the eigenvectors of $\Delta \epsilon^p$, and the aspect ratios are determined from the relations

$$w_i |_{n=1} = w_x |_{n} + w_x |_{n} (\mathbf{n}_i^{(3)} \mathbf{n}_i^{(3)} - \mathbf{n}_i^{(3)} \mathbf{n}_i^{(3)}) : \Delta \epsilon^p, \quad \alpha = 1, 2 \quad (\text{no sum over } \alpha).$$

(95)

During the calculations, if two aspect ratios are not equal but their difference is such that $|w_i - w_j|/w_i \leq 10^{-2}$, the corresponding term in Eq. (29) is omitted in order to avoid numerical difficulties, i.e., the material is assumed to be locally transversely isotropic about the axis normal to the $i$ and $j$ principal axes of the representative ellipsoid.

For comparison purposes, in some cases calculations are carried out also for the well-known Gurson model [19], in which the yield function is of the form

$$\Phi(\sigma, \overline{e}^p, f) = \left[\frac{\sigma_c}{\sigma_0(\overline{e}^p)}\right]^2 + 2f \cosh \left[\frac{3p}{2\sigma_0(\overline{e}^p)}\right] - (1 + f^2) = 0,$$

(96)

where $\sigma_c = \sqrt{3\sigma^d/2}$ is the von Mises equivalent stress, $\sigma^d = \sigma - p\overline{e}$ is the stress deviator, and $p = \sigma_{kk}/3$ is the hydrostatic stress. The corresponding “normality rule” is also used. This model assumes that voids remain spherical throughout the deformation process and the porous material is isotropic. The internal variables are now $\overline{e}^p$ and $f$, and their evolution equations are given as before by Eqs. (10) and (11). The corresponding hardening modulus is defined as

$$H = -\left(\frac{\partial \Phi}{\partial \overline{e}^p} g_1 + \frac{\partial \Phi}{\partial f} g_2\right),$$

(97)

and has dimensions of stress. The elastic part of the constitutive equations used together with the Gurson model is the same as that described in Section 2.1 with $w_1 = w_2 = 1$.

4.1. Uniaxial tension

We consider the problem of uniaxial tension of a bar made of a porous metal. In order to assess the new features of the constitutive model, the matrix material is assumed to be perfectly-plastic, i.e., the hardening exponent in (94) takes the value $n = \infty$; since the orientation of the $\mathbf{n}_i$'s does not change in this problem, any hardening or softening of the bar depends on the evolution of $(f, w_1, w_2)$, i.e., on the changes of the volume fraction and shape of the initially spherical voids. For comparison purposes, calculations are also carried out for Gurson’s model with a perfectly-plastic matrix; in the Gurson model the voids are assumed to remain spherical as the material deforms, so that the material remains isotropic, and the response depends on the evolution of porosity $f$ only.
The bar is stretched in direction 1, and the corresponding form of the deformation gradient and the stress tensor are
\[
\mathbf{F} = \lambda_a \mathbf{e}_1 \mathbf{e}_2 + \lambda_t (\mathbf{e}_2 \mathbf{e}_2 + \mathbf{e}_3 \mathbf{e}_3), \quad \text{and} \quad \mathbf{\sigma} = \mathbf{\sigma} e_1,
\]
(98)
where \((\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)\) are unit vectors along the \(x_1\)-, \(x_2\)- and \(x_3\)-axes of a fixed Cartesian frame. The value of the axial stretch ratio \(\lambda_a\) is increased gradually and the corresponding value of the transverse stretch ratio \(\lambda_t\) is determined by iteration from the condition of zero transverse stress. In the process of iteration, for every value of \(\lambda_a\) and \(\lambda_t\), the corresponding stress tensor \(\mathbf{\sigma}\) is determined numerically by using the method outlined in Section 3.

In this problem, there is no rotation of the principal Lagrangian (and Eulerian) axes and the initially spherical voids become axisymmetric ellipsoids with the longer principal axis in the direction of loading, so that
\[
\mathbf{W} = \mathbf{W}^p = \mathbf{\omega} = 0, \quad \mathbf{\sigma} = \mathbf{\sigma} = \dot{\mathbf{\sigma}} \quad \text{and} \quad \mathbf{D} = \dot{\mathbf{E}},
\]
(99)
where \(\mathbf{E}\) is the logarithmic strain tensor.

Fig. 1 shows the calculated variation of the axial stress \(\sigma\) and the hardening modulus \(H\) with the axial strain \(\epsilon = E_{11} = \ln \lambda_a\) for both the anisotropic and Gurson models. Fig. 2 shows the corresponding evolution of the porosity \(f\) and the aspect ratio of the voids in the loading direction \(w_1 (w_2 = 1 \text{ due to axial symmetry})\).

It should be noted that the hardening modulus \(H\) is proportional to and controls the sign of the slope \(d\sigma/d\epsilon^p\), where \(\epsilon^p = E_{11}^p\) is the axial plastic strain. In fact, using the flow rule \(\dot{\mathbf{E}}^p = \mathbf{D}^p = \Lambda \mathbf{N}\) together with Eq. (42) for \(\Lambda\), we conclude that
\[
\dot{\mathbf{E}}^p = \frac{1}{H} (\mathbf{N} : \dot{\mathbf{\sigma}}) \mathbf{N} = \frac{N_{11} \dot{\mathbf{\sigma}}}{H} \mathbf{N}
\]
(100)
from which follows that
\[
\frac{d\sigma}{d\epsilon^p} = \frac{H}{N_{11}}.
\]
(101)
Fig. 1a shows that, for the level of strains shown, there is very little variation of the axial stress with the axial strain $\varepsilon$ for both models. However, the slopes of these almost identical stress–strain curves are very different, i.e., there is substantial difference in the hardening/softening behavior of the two. Fig. 1b shows that the hardening modulus $H$ is small but always positive in the anisotropic model, whereas $H < 0$ in the Gurson model. The difference in the sign of $H$ can have significant effects in problems of plastic flow localization (see Section 4.3 below). A qualitative explanation of the different prediction for $H$ in the two models is given in the following.

Fig. 2a shows that both models predict an increase in porosity, with the Gurson prediction significantly higher. In the isotropic Gurson model, an increase in $f$ means that the initially spherical voids remain spherical and increase their diameter as the material is stretched. If one considers now a typical cross section perpendicular to the axis of loading, it is clear that the net load carrying area decreases for two reasons: (i) the lateral contraction of the specimen, and (ii) the increase in size of the voids intersecting the cross section. The situation is different in the anisotropic model. Fig. 2 shows that the porosity $f$ increases by only a small amount and that the voids become axisymmetric ellipsoids with the longer axis in the loading direction ($w_1 < 1$). Therefore, the “void area” on the cross-section is smaller in this case, and the void growth does not contribute much to the decrease of the load carrying area of the cross section; as a consequence, macroscopic hardening ($H > 0$) is predicted. It should be noted, however, that this particular mode of deformation of the voids in the anisotropic case weakens the material in the transverse direction more than the “isotropic” (spherical) void growth does.

4.2. The problem of simple shear

The problem analyzed in the previous section was such that there was no rotation of the principal axes of stress and strain, and the directions of orthotropy were fixed and coincident with the coordinate axes. In this section we consider the problem of simple shear and study the evolution of the axes of orthotropy. The deformation gradient is now of the form

$$F = \delta + \gamma e_1 e_2,$$  \hspace{1cm} (102)

where $\gamma$ is the amount of shearing, and $(e_1, e_2)$ the base vectors of a fixed Cartesian frame. The value of $\gamma$ is increased gradually and the constitutive equations are integrated numerically by using the method outlined.
in Section 3. Calculations are carried out for both the anisotropic and the Gurson model. A perfectly plastic matrix is assumed ($n = \infty$).

The unit vectors that define the orientation of the axes of orthotropy can be written as

$$
\mathbf{n}^{(1)} = \cos \theta \mathbf{e}_1 + \sin \theta \mathbf{e}_2, \quad \mathbf{n}^{(2)} = -\sin \theta \mathbf{e}_1 + \cos \theta \mathbf{e}_2, \quad \mathbf{n}^{(3)} = \mathbf{n}^{(1)} \times \mathbf{n}^{(2)}.
$$

The numbering of the unit vectors $\mathbf{n}^{(i)}$ is done in such a way that $\mathbf{n}^{(1)}$ and $\mathbf{n}^{(2)}$ are on the plane of deformation along the longer and shorter principal axis respectively of the local representative ellipsoid (i.e., $w_1 \leq w_2$). Therefore, the angle $\theta$ in (103) defines the direction of the longer principal axes of the local representative ellipsoid on the plane of deformation relative to the $x_1$ coordinate axis.

Fig. 3 shows the variation of the shear stress $\tau = \sigma_{12}$ and the porosity $f$ with $\gamma$ for both models. In the isotropic case (Gurson) there is no increase in porosity and the material responds as “perfectly plastic” with a constant shear flow stress that is controled by the initial porosity $f_0 = 0.04$. In the anisotropic case, both $\tau$ and $f$ decrease slightly with $\gamma$.

The variation of the angle $\theta$ and the aspect ratios $w_1$ and $w_2$ are shown in Fig. 4. The voids are elongated initially at 45°, i.e., in the direction of maximum stretching, and rotate clockwise as $\gamma$ increases. It should be noted also that the shearing direction $x_1$ is not parallel to the evolving axes of orthotropy. Therefore, nonzero normal stresses $\sigma_{11}$ and $\sigma_{22}$ develop as $\gamma$ increases; no such stresses exist in the isotropic Gurson model.

4.3. Plastic flow localization

A problem is formulated for a rectangular block of a porous metal which is constrained to plane deformations and is subjected to tension in one direction. A detailed study of this problem for incompressible materials has been given by Hill and Hutchinson [25] and Needleman [41]. The material deforms homogeneously and the initially spherical voids become ellipsoidal with the longer principal axis in the direction of stretching; at every stage of the homogeneous deformation, we examine whether a bifurcation within a localized band is possible [25,42,50]. The problem of plastic flow localization in porous media has been addressed recently also by Armero and Callari [4,6], who used finite element techniques that allow for the development of discontinuous displacement fields.
Let $x_1-x_2$ be the plane of deformation and $x_1$ the direction of stretching. The material is initially isotropic and, as it deforms plastically, becomes orthotropic with respect to the geometric $x_1-x_2-x_3$ axes. The corresponding form of the deformation gradient and the stress tensor are
\[
F = k_1 e_1 e_2 + k_2 e_2 e_2 + e_3 e_3 \quad \text{and} \quad \sigma = \sigma_1 e_1 e_1 + \sigma_3 e_3 e_3,
\]
where $(e_1, e_2, e_3)$ are unit base vectors. The condition for plastic flow localization in a shear band is that there exists a unit vector $n$ on the $x_1-x_2$ plane such that
\[
\det [n_k L^e_{kl} n_l + A_{ij}] = 0, \quad \text{where} \quad A = -\frac{1}{2}[\sigma - \sigma \cdot n n - (n \cdot \sigma \cdot n) \delta + n \cdot \sigma].
\]
If such an $n$ exists, then the direction of the shear band is perpendicular to $n$.

Since there is no rotation of the principal Lagrangian (and Eulerian) axes, the following equations hold
\[
W = W^p = \omega = 0, \quad \bar{\sigma} = \bar{\sigma} = \bar{\sigma} \quad \text{and} \quad D = \bar{E},
\]
where $\bar{E}$ is the logarithmic strain tensor. Eq. (48) can be written now as $\bar{\sigma} = \mathcal{L}^\text{ep} : \bar{E}$ where the fourth-order tensor of elastoplastic moduli has the form
\[
\mathcal{L}^\text{ep} = \mathcal{L}^e - \frac{1}{L} (\mathcal{L}^e : N)(\mathcal{L}^e : N)
\]
and has all major and minor symmetries.

Taking into account the orthotropic symmetry of the elasticity tensor $\mathcal{L}^e$ (recall that $\mathcal{L}^e$ depends on the pores and their shape evolution), one can write $\mathcal{L}^e$ in the following compact matrix form:
\[
[\mathcal{L}^e] = \begin{pmatrix}
\mathcal{L}^{e}_{1111} & \mathcal{L}^{e}_{1122} & \mathcal{L}^{e}_{1133} & 0 & 0 & 0 \\
\mathcal{L}^{e}_{1122} & \mathcal{L}^{e}_{2222} & \mathcal{L}^{e}_{2233} & 0 & 0 & 0 \\
\mathcal{L}^{e}_{1133} & \mathcal{L}^{e}_{2233} & \mathcal{L}^{e}_{3333} & 0 & 0 & 0 \\
0 & 0 & 0 & G_{12} & 0 & 0 \\
0 & 0 & 0 & 0 & G_{13} & 0 \\
0 & 0 & 0 & 0 & 0 & G_{23}
\end{pmatrix},
\]
where $G_{12} \equiv \mathcal{L}^{e}_{1212}$, $G_{13} \equiv \mathcal{L}^{e}_{1313}$ and $G_{23} \equiv \mathcal{L}^{e}_{2323}$ are the elastic shear moduli.
Taking into account that the homogeneous solution does not involve shearing relative to the coordinate axes, so that \( \mathbf{N} = N_{11} \mathbf{e}_1 \mathbf{e}_1 + N_{22} \mathbf{e}_2 \mathbf{e}_2 + N_{33} \mathbf{e}_3 \mathbf{e}_3 \), we conclude that the tensor of the elastoplastic moduli can be written compactly as

\[
[\mathcal{C}^{ep}] = \begin{pmatrix}
\mathcal{C}^{ep}_{1111} & \mathcal{C}^{ep}_{1122} & \mathcal{C}^{ep}_{1133} & 0 & 0 & 0 \\
\mathcal{C}^{ep}_{1122} & \mathcal{C}^{ep}_{2222} & \mathcal{C}^{ep}_{2233} & 0 & 0 & 0 \\
\mathcal{C}^{ep}_{1133} & \mathcal{C}^{ep}_{2233} & \mathcal{C}^{ep}_{3333} & 0 & 0 & 0 \\
0 & 0 & 0 & G_{12} & 0 & 0 \\
0 & 0 & 0 & 0 & G_{13} & 0 \\
0 & 0 & 0 & 0 & 0 & G_{23}
\end{pmatrix}.
\] (109)

Finally, carrying out the algebraic manipulations in (105), we find that the localization condition can be written as

\[
\det[\mathcal{B}_{ij}] = 0,
\] (110)

where

\[
\begin{align*}
B_{11} &= \mathcal{C}^{ep}_{1111} n_1^2 + \left( G_{12} - \frac{\sigma_2}{2} \right) n_2^2, \\
B_{12} &= \mathcal{C}^{ep}_{1122} + G_{12} + \frac{\sigma_1}{2} n_1 n_2, \\
B_{21} &= \mathcal{C}^{ep}_{1122} + G_{12} - \frac{\sigma_1}{2} n_1 n_2, \\
B_{22} &= \mathcal{C}^{ep}_{2222} n_1 n_2 + \left( G_{12} - \frac{\sigma_1}{2} \right) n_1^2, \\
B_{33} &= G_{13} + \frac{\sigma_1}{2} n_1^2 + G_{23} n_2^2 - \frac{\sigma_3}{2}
\end{align*}
\] (111)

and \( B_{13} = B_{23} = B_{31} = B_{32} = 0 \). Since the stress components \( \sigma_1 \) and \( \sigma_3 \) are of order \( \sigma_0 \), which is several orders of magnitude smaller than the elastic moduli \( G_{ij} \), the component \( B_{33} \) is always positive and the localization condition (110) can be written as

\[
B_{11} B_{22} - B_{12} B_{21} = 0.
\] (112)

The calculation of the stage at which localization of plastic flow is possible is carried out numerically as described in the following. The homogeneous solution is determined numerically by increasing gradually the axial stretch ratio \( \lambda_1 \) and determining the corresponding value of the transverse stretch ration \( \lambda_2 \) by iteration from the condition of zero transverse stress, i.e., \( \mathbf{e}_2 \cdot \mathbf{t} = 0 \). In the process of iteration, for every value of \( \lambda_1 \) and \( \lambda_2 \), the corresponding stress tensor \( \mathbf{t} \) is determined numerically by using the method outlined in Section 3. Once the homogeneous solution has been determined, we set \( \mathbf{n} = \cos \psi \mathbf{e}_1 + \sin \psi \mathbf{e}_2 \) and the localization condition (112) is examined by scanning the range \( 0^\circ \leq \psi < 90^\circ \); if a change of sign of the quantity \( B_{11} B_{22} - B_{12} B_{21} \) is detected, the corresponding root that defines the localization angle \( \psi \) is determined.

Calculations are carried out for values of the hardening exponent \( \eta = \infty \) and \( \eta = 10 \). For comparison purposes, calculations are also carried out for Gurson’s model and the results are summarized in Tables 1 and 2.

<table>
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<th>( \mathbf{n} )</th>
<th>( \mathbf{n} )</th>
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<th>( \mathbf{n} )</th>
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<td>1.091</td>
<td>0.0404</td>
<td>0.988</td>
<td>1.009</td>
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</tr>
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<td>1.0</td>
<td>1.0</td>
<td>-0.27\sigma_0</td>
<td>44.0^\circ</td>
</tr>
</tbody>
</table>

Table 1
Localization conditions for \( \eta = \infty \)
In the case of the perfectly plastic matrix \( n = 1 \) localization is predicted at about the same strain for both models. Note that in plane-strain tension (as opposed to uniaxial tension) and for \( n = 1 \), both the anisotropic and the Gurson models develop a negative plastic modulus \( H \) once the material deforms plastically. The situation is entirely different for the case of the hardening matrix with \( n = 10 \). In the anisotropic model the localization condition is never met for values of the axial strain up to \( \epsilon = \ln \lambda_1 = 2.3979 \) (i.e., \( \lambda_1 = 11 \)), where the calculations are terminated; the value of \( H \) is positive and decreases with \( \epsilon \), but never becomes small enough to cause localization in this range of strains. In the Gurson model, the \( H \) values decrease faster with \( \epsilon \) and localization is predicted at a strain level \( \epsilon = \ln \lambda_1 = 0.5593 \) (i.e., \( \lambda_1 = 1.749 \)).

### 4.4. Necking bifurcation in plane-strain tension

We analyze again the problem of plane-strain tension described in Section 4.3 and look for possible bifurcations. We consider a rectangular block with aspect ratio \( L_0/B_0 = 3 \), where \( 2L_0 \) is the length of the specimen and \( 2B_0 \) its width. We introduce the Cartesian system shown in Fig. 5 and identify each material...
particle in the specimen by its position vector \( \mathbf{X} = (X_1, X_2) \) in the undeformed configuration. We are interested in symmetric solutions and consider one quarter of the full block as shown in Fig. 5. The deformation is driven by the uniform prescribed end displacement \( \ddot{u} \) in the \( X_2 \)-direction on the shear-free end \( X_2 = L_0 \); the lateral surface on \( X_1 = B_0 \) is kept traction free.

A “perfect” specimen is analyzed first and bifurcation analysis is carried out as described in the following. In a separate set of calculations, a geometric imperfection is introduced in the specimen and the development of the “neck” is studied. The analyses are carried out by using the finite element method. Four-node isoparametric plane-strain elements with 2 × 2 Gauss integration stations and the so-called B-bar method (Hughes [26,27]) are used in the computation.

**Remark 8.** Since the porous material is compressible, use of the B-bar method is not a requirement. However, mesh-size convergence studies of the bifurcation point show that convergence is substantially faster when B-bar elements are used.

Needleman [41] and Tvergaard [51,52] used the ideas introduced by Hill [21–23] and developed a method for the calculation of an upper bound to the bifurcation load by using the so-called “linear comparison solid” (referred to as “LCS” in the following), which is characterized by a constitutive equation of the form

\[
\hat{\sigma} = \mathcal{L}^{\text{LCS}} : \mathbf{D},
\]

where \( \mathcal{L}^{\text{LCS}} \) is equal to the plastic branch \( \mathcal{L}^{\text{epl}} \) of the elastic–plastic moduli for every material point currently on the yield surface, and the elastic branch \( \mathcal{L}^{\text{e}} \) elsewhere; i.e., the relationship \( \hat{\sigma} \cdot \mathbf{D} \) of the LCS is linear and independent of the “loading–unloading” criterion. The calculated bifurcation point for the LCS is also a possible bifurcation for the actual elastic–plastic material; however, earlier bifurcations for the actual solid cannot be ruled out and, in that sense, the bifurcation point of the LCS provides an upper bound for the possible bifurcation for the actual elastic–plastic material. At some critical extension, say \( \ddot{u}_c \), a bifurcation from the fundamental homogeneous solution first becomes possible. Then, a nontrivial solution \( \mathbf{v} \) exists for the following variational problem (see also [37]):

\[
\int_V \left[ \mathbf{D}^* : \mathcal{L}^{\text{LCS}} : \mathbf{D} + D_{kl} \sigma : \mathbf{D} - \sigma : (2\mathbf{D} \cdot \mathbf{D} - \mathbf{L}^T \cdot \mathbf{L}) \right] \, dV = 0 \quad \forall \mathbf{v} \in A,
\]

where \( A \equiv \{ \mathbf{v} | v_2 = 0 \text{ on } X_2 = 0 \text{ and } X_1 = L_0, \quad v_1 = 0 \text{ on } X_1 = 0 \} \). \( \mathbf{L} = \dot{\mathbf{v}} \) is the velocity gradient in the undeformed configuration and \( \mathbf{D} \) its symmetric part; similarly, \( \mathbf{L}^* = \mathbf{v} \cdot \nabla \) and \( \mathbf{D}^* = (\mathbf{L}^* + \mathbf{L}^{*T})/2 \).

The bifurcation calculations for the LCS are carried out in a way similar to that used by Needleman [39]. The finite element solution for the homogeneous problem (fundamental solution) is determined incrementally by increasing gradually the imposed displacement \( \ddot{u} \); at every step of the calculation it is examined whether the discrete problem resulting from (114)

\[
[K(\ddot{u})] \{ v^N \} = \{ 0 \},
\]

has a nontrivial solution for \( \{ v^N \} \). In the above equation \( [K(\ddot{u})] \) is the “stiffness matrix” resulting from (114) and \( \{ v^N \} \) is the corresponding “vector of nodal velocities”.

Let \( v^C \) be the normalized eigenmode corresponding (114) (or (115) in discrete form) at the critical value \( \ddot{u}_c \); then the complete solution to the incremental boundary value problem is

\[
\mathbf{v} = v^F + \zeta v^C,
\]

where \( v^F \) is the fundamental solution, and the amplitude \( \zeta \) is chosen so that plastic loading occurs everywhere in the current plastic zone, except at one point where neutral load takes place [28,29].

A value of the hardening exponent \( n = 10 \) is used in the calculations. For the anisotropic model bifurcation in the LCS becomes first possible at a strain level
\[ \epsilon_c = \ln \left( 1 + \frac{\dot{u}_c}{L_0} \right) = 0.1484. \quad (117) \]

The corresponding value for the LCS in the Gurson material is smaller: \( \epsilon_c = 0.1249. \) The state of the homogeneous solution at the bifurcation point is summarized in Table 3 for both materials.

The corresponding normalized eigenmode \( v^c \) for the anisotropic model calculated by using a 15\times45 finite element mesh is shown in Fig. 6. Clearly, \( v^c \) corresponds to a necking mode in the specimen.

In order to simplify the calculation of the neck development, a geometric imperfection was introduced in the undeformed configuration, and the “imperfect” specimen was analyzed for the anisotropic and Gurson models by using the finite element method. In particular, the undeformed configuration of the specimen was perturbed in a way resembling the necking mode, i.e., the initial width of the specimen was assumed to vary in the \( X_2 \) direction according to the formula

\[ \tilde{B}_0(X_2) = B_0 - \zeta B_0 \cos \left( \frac{\pi X_2}{2 L_0} \right), \quad (118) \]

where the value \( \zeta = 0.005 \) was used. In this case, the neck develops gradually and the first elastic unloading appears at the upper-left corner of the mesh shown in Fig. 7 at a strain level \( \epsilon = \ln(1 + \dot{u}/L_0) = 0.1178 \) for the anisotropic model. The corresponding value for the Gurson model is \( \epsilon = 0.1118. \)

Table 3

<table>
<thead>
<tr>
<th></th>
<th>( \epsilon_c )</th>
<th>( \sigma_c/\sigma_0 )</th>
<th>( f_c )</th>
<th>( w_{1c} )</th>
<th>( w_{2c} )</th>
<th>( H_c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Anisotropic</td>
<td>0.1484</td>
<td>1.586</td>
<td>0.0473</td>
<td>0.812</td>
<td>1.240</td>
<td>7.93\sigma_0</td>
</tr>
<tr>
<td>Gurson</td>
<td>0.1249</td>
<td>1.568</td>
<td>0.0480</td>
<td>1.0</td>
<td>1.0</td>
<td>1.49\sigma_0</td>
</tr>
</tbody>
</table>

Fig. 6. Normalized bifurcation eigenmode.
Fig. 7 shows the plastic zones (yellow regions) at strain level $\varepsilon = 0.1823$.

Fig. 8 shows the contours of porosity at a strain level $\varepsilon = 0.1823$.

4.5. Ductile fracture

We consider the problem of a plane-strain mode-I blunt crack in a homogeneous porous elastoplastic material under small scale yielding conditions. A boundary layer formulation is used in order to study the
near-tip stress and deformation fields. Traction free boundary conditions are used on the crack face and
displacement boundary conditions remote from the tip are applied incrementally to impose an asymptotic
dependence on the mode-I elastic solution, i.e.,

\[
\begin{bmatrix}
  u_1 \\
  u_2
\end{bmatrix} = \frac{K_I}{2\mu} \sqrt{\frac{r}{2\pi}} (3 - 4\nu - \cos \theta) \begin{bmatrix}
  \cos(\theta/2) \\
  \sin(\theta/2)
\end{bmatrix} + \begin{bmatrix}
  c \\
  0
\end{bmatrix},
\]

(119)

where \( u_i \) are the displacement components, \( K_I \) is the mode-I stress intensity factor, \( x_1 \) and \( x_2 \) are crack-tip Cartesian coordinates with the \( x_1 \) being the axis of symmetry and \( x_2 \) the direction of mode-I loading, and \( (r, \theta) \) are crack-tip polar coordinates. The constant \( c \) in the above equation represents a rigid body
translation in the \( x_1 \)-direction, and is determined so that the \( u_1 \) component of the displacement at the
outermost point on the \( x_1 \)-axis vanishes.

Four-node elements similar to those discussed in Section 4.4 are used in the calculations. The outermost
radius of the finite element mesh, where the elastic asymptotic displacement field (119) is imposed, is
\( R \approx 1.2 \times 10^3 b_0 \), where \( b_0 \) is the initial radius of the semicircular notch at the tip of the blunt crack. Because
of symmetry, only half of the region (i.e., \( 0 \leq \theta \leq \pi \)) is analyzed. The finite element mesh in the region near
the crack tip is shown in Fig. 9. A total of 1658 elements are used in the computations.

The value \( n = 10 \) for the hardening exponent of the matrix is used in the calculations, which are carried
out for both the anisotropic and the Gurson models.

In this set of calculations we take into account the nucleation of new voids in the material by cracking or
interfacial decohesion of inclusion or precipitate particles. The evolution equation of porosity (initially set
to \( f_0 = 0.04 \)) is now written as

\[
\dot{f} = (1 - f) D^{p}_k + \mathcal{A} \dot{\varepsilon}^p,
\]

(120)

where the \( \mathcal{A} \)-term accounts for the aforementioned void nucleation. The parameter \( \mathcal{A} \) is chosen so that the
nucleation strain follows a normal distribution with mean value \( \varepsilon_N \) and standard deviation \( s_N \) [8]:

\[
\mathcal{A}(\varepsilon^p) = \frac{f_N}{s_N \sqrt{2\pi}} \exp \left[ -\frac{1}{2} \left( \frac{\varepsilon^p - \varepsilon_N}{s_N} \right)^2 \right],
\]

(121)

where \( f_N \) is the volume fraction of void nucleating particles. The values \( f_N = 0.04, \varepsilon_N = 0.4 \) and \( s_N = 0.1 \) are
used in the computations. Eq. (67) is replaced now by

![Fig. 9. The finite element mesh in the region near the blunt crack tip.](image)
\[ f_{n+1}(\Delta \varepsilon^p) = f_n + (1 - f_n) \Delta \varepsilon_{kk}^p + \mathcal{A}(\varepsilon_n^p) \Delta \varepsilon^p(\Delta \varepsilon^p), \]

where \( \Delta \varepsilon^p(\Delta \varepsilon^p) = \sigma_n : \Delta \varepsilon^p / [(1 - f_n)\sigma_{yy}(\varepsilon_n^p)] \).

Fig. 10 shows the deformed finite element mesh superposed on the undeformed (dash lines) at a load level \( K_1 \equiv K_i / (\sigma_0\sqrt{b_0}) = 30 \) for the case of the anisotropic model.

Figs. 11–14 show the variation of the opening stress \( \sigma_{22} \), the hydrostatic stress \( p = \sigma_{kk}/3 \), the porosity \( f \), and the equivalent plastic strain of \( \varepsilon^p \), ahead of the crack tip at different load levels for both the anisotropic and Gurson models. In these figures, “\( x \)” is the distance of a material point in the undeformed configuration from the root of the semicircular notch.

For the case of the Gurson model, the computations cannot be continued for values of \( K_1 \equiv K_i / (\sigma_0\sqrt{b_0}) \) beyond 36.11, because the porosity \( f \) takes very high values ahead of the crack tip, the local load carrying...
capacity of the material decreases substantially, and the overall equilibrium Newton scheme does not converge.

The prediction of the development of “damage” ahead of the crack is substantially different for the anisotropic and Gurson models. Starting with the anisotropic material, we notice that the opening stress $\sigma_{22}$ at all material points ahead of the crack increases monotonically with load (Fig. 11a). Since the blunt crack tip is traction-free, the maximum of $p$ and $\sigma_{22}$ appear at some distance ahead of the crack tip. Figs. 13a and 14a show that the porosity $f$ and the equivalent plastic strain $\bar{\varepsilon}^p$ in the anisotropic material take a maximum.
value at the root of the blunt crack and become progressively smaller ahead of the crack. Therefore, the anisotropic material models would suggest that the crack extension would initiate at the root of the blunt crack tip.

The situation is entirely different for the isotropic Gurson model. The opening stress $\sigma_{22}$ ahead of the crack tip increases initially, but, after a while, an “island” of reduced load-carrying capacity is formed ahead of the crack tip and $\sigma_{22}$ drops locally (Fig. 11b). This is due to the rapid porosity increase in that region as explained in the following. The evolution of porosity is given by $f = (1 - f)D_{kk} + \mathcal{S}\dot{\varepsilon}^p$, so that

$$\dot{f} = 2\mu\dot{\varepsilon} : \mathbf{m}(s) : \sigma + \mathcal{S}\dot{\varepsilon}^p$$

for the anisotropic, and

$$\dot{f} = \frac{3A}{\sigma_y}f(1 - f)\sinh\left(\frac{3p}{2\sigma_y}\right) + \mathcal{S}\dot{\varepsilon}^p$$

for the Gurson model.

The last two equations show that $\dot{f}$ depends linearly on $\sigma$ in the anisotropic model and exponentially on $p$ in the Gurson model. As mentioned before, the hydrostatic stress in the crack tip region reaches its maximum value at some distance ahead of the crack root. Therefore, the rate of porosity growth at the point of maximum $p$ ahead of the crack predicted by the Gurson model is much faster than that predicted by the anisotropic model, as shown in Fig. 13 (note the different porosity scales in Fig. 13a and b). The exponential pressure term in (124) dominates, and a region of high porosity is created ahead of the crack in the Gurson material. As a consequence, the value of $\sigma_{22}$ drops locally with a simultaneous local increase of the equivalent plastic strain $\varepsilon^p$ at the “weak” region of high porosity (Figs. 11b and 14b for $K1 = 36.11$). Therefore, the predicted mechanism of crack extension in the Gurson material is the formation of a new crack at some distance ahead of the blunt tip of the original crack; this new crack eventually grows towards the major crack, leading to macroscopic crack extension.

Figs. 15 and 16 show contours of the equivalent plastic strain $\varepsilon^p$ in the region near the blunt crack tip for the anisotropic and Gurson models for a load level of $K1 = 36.11$. The qualitative difference of the distribution of $\varepsilon^p$ ahead of the crack in the two materials reflects the aforementioned different prediction of crack extension mechanism.

Figs. 17 and 18 show contours of the aspect ratios $w_1$ and $w_2$ in the region near the blunt crack tip for the anisotropic model. In the anisotropic material, the initially spherical voids become elongated ellipsoids in the crack tip region with their longer axis in the direction of macroscopic loading $x_2$. 
It should be noted that the behavior of the spherical voids ahead of the crack tip is substantially different from the behavior of isolated long cylindrical voids parallel to the crack front (long cylindrical voids perpendicular to the $x_1-x_2$ plane). Aravas and McMeeking [2,3] carried out detailed calculations for the behavior of such isolated cylindrical holes ahead of a mode-I crack under conditions of plane strain using...
either the von Mises or the Gurson model to describe the constitutive behavior of the “matrix” material; their results show that such holes are pulled towards the crack tip and change their shape to approximately

$$K1 \equiv \frac{K1}{(\sigma_0 \sqrt{h_0})} = 36.11.$$
elliptical with the major axis radial to the crack. Aravas and McMeeking used the results of the afore-
mentioned calculations together with several crack–hole coalescence criteria and obtained estimates for the
value of the crack-tip-opening-displacement (COD) for fracture initiation; these estimates underestimate
the critical COD obtained from fracture toughness test [2,3,43]. The present calculations with the aniso-
tropic model provide a possible explanation for this underestimation. The results presented earlier make it
clear that the crack–void interaction is not as strong for the case of spherical voids and that the use of
isotropic models, such as the Gurson model, may overestimate the rate of porosity growth, thus under-
estimating the fracture initiation load. It appears that more realistic estimates for the conditions of ductile
fracture initiation can be obtained when the anisotropic model is combined with a local failure criterion in
the near-tip region; such calculations are now underway.

Finally, it should be emphasized that further work needs to be carried out before definitive conclusions
can be drawn concerning the validity of the various predictions of the anisotropic model relative to those of
the earlier Gurson model. While, as we have seen, the additional degrees of freedom allowed by the
anisotropic model—in terms of allowing void shape and orientation changes—are generally expected to
lead to improved predictions over any isotropic model ignoring such effects, it should be kept in mind that
the anisotropic model also has limitations of its own, including most critically for this particular appli-
cation, the fact that it tends to give overly high estimates for the instantaneous yield stress under high-
triaxiality conditions, where the Gurson model is known to be very accurate [48]. It is conceivable that the
significant differences between the overall predictions of the two models for the blunting crack problem
could be due to the differences between the yield predictions of the two models for large triaxiality con-
ditions (which are known to hold in the neighborhood of the blunted crack tip). However, our experience
with other (admittedly simpler) problems [48,35] would tend to suggest that the qualitative differences
observed in the context of the blunting crack problem could not be explained by the relatively small (i.e.,
quantitative) differences in the effective yield predictions of the two models for high-triaxiality conditions.
Instead, these qualitative differences should be a consequence of the richer physics described by the
anisotropic model, which can account for the developing anisotropy in the damage evolution process taking
place near the tip of the blunting crack.

Acknowledgements

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Availability of computer program: Copies of the computer code (UMAT) will be supplied upon request.
Please address inquiries to Prof. Aravas at the e-mail address aravas@uth.gr.

Appendix A. The Eshelby tensors

The Eshelby tensors $S$ and $\Pi$ are evaluated as follows. Let $\mathcal{M}$ be the compliance tensor of the elastic
matrix. Willis [53,55] has shown that the tensors $P = S : \mathcal{M}$ and $R = \Pi : \mathcal{M}$ can be written in the form

$$P = \frac{1}{4\pi w_1 w_2} \int_{|\xi|=1} H(\xi) \frac{dS(\xi)}{|Z^{-1} \cdot \xi|^3}, \quad R = \frac{1}{4\pi w_1 w_2} \int_{|\xi|=1} \hat{H}(\xi) \frac{dS(\xi)}{|Z^{-1} \cdot \xi|^3},$$

(A.1)

where

$$H_{ijkl}(\xi) = \left[ \mathcal{L}^{-1}_2(\xi) \right]_{ik} \xi_j \xi_l, \quad \hat{H}_{ijkl}(\xi) = \left[ \mathcal{L}^{-1}_2(\xi) \right]_{ik} \xi_j \xi_l,$$

(A.2)

$$\mathcal{L}^{-1}_2(\xi)_{ik} = \mathcal{L}^{-1}_2(\xi)_{ik}, \quad Z = w_1 n^{(1)} + w_2 n^{(2)} + n^{(3)},$$

(A.3)
and the notation
\[
A_{(i)(j)} = \frac{1}{4}(A_{ijkl} + A_{ijlk} + A_{ikjl} + A_{jikl}),
\]
(A.4)

and
\[
A_{|i|} = \frac{1}{4}(A_{ijkl} + A_{ijlk} - A_{ikjl} - A_{jikl})
\]
(A.5)
is used. Eqs. (A.1) imply that

\[
\text{and } \int_{|\xi|=1} \mathbf{H}(\xi) : \mathbf{dS}(\xi) = \frac{1}{4\pi w_1 w_2} \int_{|\xi|=1} \mathbf{H}(\xi) : \mathbf{dS}(\xi) / |Z^{-1} \cdot \xi|^3;
\]

\[
\Pi = \frac{1}{4\pi w_1 w_2} \int_{|\xi|=1} \hat{\mathbf{H}}(\xi) : \mathbf{dS}(\xi) / |Z^{-1} \cdot \xi|^3.
\]

(A.6)

For isotropic materials \( \mathbf{L} = 2\mu \mathbf{K} + 3\kappa \mathbf{J} \), so that
\[
\mathbf{L}_2(\xi) = \mu \left( |\xi|^2 \delta + \frac{1}{1 - 2\nu} \xi^2 \right) \quad \text{and} \quad \mathbf{L}_2^{-1}(\xi) = \frac{1}{2\mu |\xi|^4} \left[ |\xi|^2 \delta - \frac{1}{2(1 - 2\nu)} \xi^2 \right].
\]

(A.7)

Using Eqs. (A.2) and (A.7), we find after some algebra that
\[
\left( \mathbf{H}(\xi) : \mathbf{L} \right)_{ijkl}(\xi, \nu) = \frac{1}{2|\xi|^2} \left( \delta_{ik} \xi_j \xi_l + \delta_{jk} \xi_i \xi_l + \delta_{ij} \xi_k \xi_l + \delta_{jl} \xi_i \xi_k \right)
\]
\[
- \frac{1}{|\xi|^4} \frac{1}{1 - \nu} \xi_i \xi_j \xi_k \xi_l + \frac{1}{|\xi|^2} \frac{\nu}{1 - \nu} \xi_i \xi_j \delta_{kl}
\]

(A.8)

and
\[
\left( \hat{\mathbf{H}}(\xi) : \mathbf{L} \right)_{ijkl}(\xi) = \frac{1}{2|\xi|^2} \left( \delta_{ik} \xi_j \xi_l - \delta_{jk} \xi_i \xi_l + \delta_{ij} \xi_k \xi_l - \delta_{jl} \xi_i \xi_k \right).
\]

(A.9)

Kailasam et al. [35] have shown that \( \mathbf{Q} \) can be written in the form
\[
\frac{1}{\mu} \mathbf{Q} = \frac{1}{4\pi w_1 w_2} \int_{|\xi|=1} \mathbf{E}(\xi) : \mathbf{dS}(\xi) / |Z^{-1} \cdot \xi|^3.
\]

(A.10)

where
\[
\mathbf{E}_{ijkl}(\xi, \nu) = \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} - \frac{1}{|\xi|^2} \left( \delta_{ik} \xi_j \xi_l + \delta_{jl} \xi_i \xi_k + \delta_{ij} \xi_k \xi_l + \delta_{kl} \xi_i \xi_j \right)
\]
\[
+ \frac{2\nu}{1 - \nu} \left[ \delta_{ij} \delta_{kl} - \frac{1}{|\xi|^2} \left( \xi_i \xi_j \xi_k \xi_l + \delta_{kl} \xi_i \xi_j \right) \right] + \frac{2}{|\xi|^4} \frac{1}{1 - \nu} \xi_i \xi_j \xi_k \xi_l.
\]

(A.11)

Initially, the integral definitions (A.6) and (A.10) were used for the numerical evaluation of \( \mathbf{S}, \mathbf{II} \) and \( \mathbf{Q} \). In all three cases, one needs to evaluate numerically integrals of the form
\[
I = \int_{|\xi|=1} A(\xi) \, dS(\xi) = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} A(\xi(\theta, \phi)) \sin \phi \, d\theta \, d\phi,
\]

(A.12)

where \( \xi = (\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi) \). After the transformations \( \theta(r) = (r + 1)\pi \) and \( \phi(s) = (s + 1)\pi/2 \), the above integral becomes
\[
I = \frac{\pi^2}{2} \int_{-1}^{1} \int_{-1}^{1} A(\xi(\theta(r), \phi(s))) \sin \phi(s) \, dr \, ds,
\]

(A.13)
and can be evaluated numerically by using Gauss integration of the form

\[ I \cong \frac{\pi^2}{2} \sum_{i=1}^{NG} \sum_{j=1}^{NG} W_i W_j A(\xi_i, \phi_j) \sin \phi_j, \quad (A.14) \]

where \( \theta = \theta(r_i) \) and \( \phi_j = \phi(s_j) \), \( r_i \) and \( s_j \) are the integration stations, \( W_i \) and \( W_j \) the corresponding weights, and \( NG \) the number of Gauss integration stations. A similar method for the evaluation of \( \mathbf{S} \) has been proposed by Ghahremani [15].

It was soon realized though that, when at least one of the aspect ratios \( w_1 \) and \( w_2 \) was substantially different from unity, a large number of Gauss points are required for accurate evaluation of \( \mathbf{S} \), \( \Pi \) and \( \mathbf{Q} \). For example, for the case of \( w_1 = 0.2 \) and \( w_2 = 5 \), a number of \( NG = 257 \) Gauss points is required in each direction for a less that 1% error in the components of \( \mathbf{S} \) (similar findings are reported also by Gavazzi and Lagoudas [14]); since the evaluation of these tensors is carried out repeatedly at each Gauss integration point in the finite element mesh, the finite element calculations become extremely time consuming. In order to accelerate the calculations, the original expressions for the components of \( \mathbf{S} \) and \( \Pi \) relative to a coordinate system aligned with the axes of the ellipsoid as given by Eshelby [10,11] were used. These calculations require only the numerical evaluation of elliptic integrals, are much faster, and provide the components \( S_{ijkl} \) and \( \Pi_{ijkl} \) with respect to axes aligned with the \( n^{(i)} \)'s; the corresponding components \( S_{ijkl}^{*} \) and \( \Pi_{ijkl}^{*} \) relative to the fixed coordinate axes used in the finite element calculations are obtained by using standard tensor transformation formulae.

In the following, we use Eqs. (A.10) and (A.11) and derive analytic expressions for the corresponding components \( Q_{ijkl} \) of \( \mathbf{Q} \). Let \( a_1 = a_3/w_1 \), \( a_2 = a_3/w_2 \) with \( a_1 = 1 \), and arrange the numbering so that \( a_1 \geq a_2 \geq a_3 \). Using (A.11), we can show that the nonzero components \( Q_{ijkl} \) of \( \mathbf{Q} \) relative to the local coordinate system defined by the \( n^{(i)} \)'s, and for \( a \equiv a_1 > b \equiv a_2 > c \equiv a_3 \) are

\[ Q_{1111} = \frac{\mu}{8\pi(1-v)} \left[ 8\pi - I_a - 3a^2 I_{aa} \right], \quad (A.15) \]

\[ Q_{1122} = \frac{\mu}{8\pi(1-v)} \left[ 16\pi v + (1 - 4v)(I_a + I_b) - 3(a^2 + b^2)I_{ab} \right], \quad (A.16) \]

\[ Q_{1212} = \frac{\mu}{8\pi(1-v)} \left[ 8\pi(1-v) - (1 - 2v)(I_a + I_b) - 3(a^2 + b^2)I_{ab} \right], \quad (A.17) \]

where

\[ I_a = \frac{4\pi a b c}{(a^2 - b^2)\sqrt{a^2 - c^2}} \left[ F(\theta, k) - E(\theta, k) \right], \quad (A.18) \]

\[ I_c = \frac{4\pi a b c}{(b^2 - c^2)\sqrt{a^2 - c^2}} \left[ \frac{b\sqrt{a^2 - c^2}}{ac} - E(\theta, k) \right], \quad (A.19) \]

\[ I_b = 4\pi - I_a - I_c, \quad (A.20) \]

\[ I_{ab} = \frac{I_b - I_a}{3(a^2 - b^2)}, \quad I_{ac} = \frac{I_c - I_a}{3(a^2 - c^2)}, \quad I_{aa} = \frac{4\pi}{3a^2} - I_{ab} - I_{ac}, \quad (A.21) \]

\[ \theta = \sin^{-1} \sqrt{1 - \frac{a^2}{c^2}}, \quad k = \sqrt{\frac{a^2 - b^2}{a^2 - c^2}} \quad (A.22) \]
and

\[ F(\theta, k) = \int_0^\theta \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}, \quad E(\theta, k) = \int_0^\theta \sqrt{1 - k^2 \sin^2 \phi} \, d\phi \]  

(A.23)

are the elliptic integrals of the first and second kinds. All other nonzero components \( Q'_{ijkl} \) are found by simultaneous cyclic interchange of \( (1, 2, 3) \) and \( (a, b, c) \). In the special cases where \( a = b > c \) or \( a > b = c \), the quantities \( I_a, I_b, I_c, I_{ab}, I_{ac}, \) and \( I_{bc} \) take the values given on page 385 of Eshelby’s original article [10]. The special case of isotropy \( a = b = c \) is discussed in Appendix B that follows.

Appendix B. The isotropic case

For the special case of spherical voids \( (w_1 = w_2 = 1) \) the tensors \( S, \Pi \) and \( Q \) take the form

\[ S(\nu) = \frac{2}{15} \left( \frac{4 - 5\nu}{1 - \nu} K + \frac{1 + \nu}{3} \frac{1}{1 - \nu} J \right), \quad \Pi = 0, \]  

(B.1)

\[ Q(\mu, \nu) = 2\mu \left( \frac{1}{15} \frac{7 - 5\nu}{1 - \nu} K + \frac{2}{3} \frac{1 + \nu}{1 - \nu} J \right), \]  

(B.2)

so that

\[ Q^{-1}(\mu, \nu) = \frac{1}{2\mu} \left( 15 \frac{1 - \nu}{7 - 5\nu} K + \frac{3}{2} \frac{1 - \nu}{1 + \nu} J \right). \]  

(B.3)

In that case, the effective compliance tensor \( \hat{C} = C + \frac{f}{1 - f} Q^{-1} \) takes the form

\[ \hat{C} = \frac{1}{2\mu} K + \frac{1}{3\kappa} J, \quad \hat{\mu} = \mu, \quad \frac{1 - f}{1 + 2f} \frac{24 - 5\nu}{7 - 5\nu}, \quad \hat{\kappa} = \kappa, \frac{1 - f}{1 + f} \frac{1 + \nu}{2(1 - 2\nu)}, \]  

(B.4)

where \( \kappa = 2\mu(1 + \nu)/[3(1 - 2\nu)] \) is the bulk modulus of the matrix. The corresponding effective Young’s modulus is \( \tilde{E} = \frac{\kappa}{(3\kappa + \hat{\mu})} \), i.e.,

\[ \tilde{E} = E \frac{1 - f}{1 + \frac{(1 + \nu)(13 - 15\nu)}{2(1 - 2\nu)}}, \]  

(B.5)

where \( E = 2\mu(1 + \nu) \) is the Young modulus of the matrix.

Also, the tensors \( A^{-1} = I - (1 - f)S |_{v=1/2} \) and \( C = (1 - f) \Pi : A \) take the form

\[ A^{-1}(f) = 3 + \frac{2f}{5} K + f J, \quad A(f) = \frac{5}{3 + 2f} K + \frac{1}{f} J, \quad C = 0. \]  

(B.6)

The tensor \( m(f) = 3\mu A^{-1} |_{v=1/2} \) that enters the yield function also takes the form

\[ m(f) = \frac{1}{2} \left( \frac{3}{1 - f} + \frac{2f}{5} K + \frac{3}{4} \frac{f}{1 - f} J \right), \]  

(B.7)

so that the yield condition becomes

\[ \Phi(\sigma, \varepsilon, f) = \left( 1 + \frac{2}{3f} \right) \left( \frac{\sigma_e}{1 - f} \right)^2 + \left( \frac{9}{4f} \frac{\varepsilon}{1 - f} \right)^2 - \sigma_e^2(\varepsilon^p) = 0, \]  

(B.8)

where \( \sigma_e = \sqrt{3\sigma^d : \sigma^d/3} \) is the von Mises equivalent stress, \( \sigma^d = \sigma - p\hat{\sigma} \) is the stress deviator, and \( p = \sigma_{kk}/3 \) is the hydrostatic stress.
Using the above yield criterion for a state of uniaxial tension \( \sigma \) \( (\sigma_z = \sigma, p = \sigma/3) \), we conclude that the predicted flow stress of the porous metal \( \bar{\sigma}_0 \) is

\[
\bar{\sigma}_0 = \frac{1 - f}{\sqrt{1 + \frac{11}{12} f}} \sigma_0,
\]

where \( \sigma_0 \) is the flow stress of the matrix. The corresponding flow stress in shear \( \bar{\tau}_0 \) is

\[
\bar{\tau}_0 = \frac{1 - f}{\sqrt{1 + \frac{5}{3} f}} \frac{\sigma_0}{\sqrt{3}}.
\]

**Appendix C. The consistency condition**

The yield function is of the form

\[
\Phi(\sigma, s) = \frac{1}{1 - f} \sigma : \mathbf{m}(s) : \sigma - \sigma_0^2(\varepsilon^p) = 0,
\]

where

\[
\mathbf{m}(s) = \frac{3}{2} \mathbf{K} + \frac{3f}{1 - f} \mu \mathbf{Q}^{-1}|_{r=1/2}.
\]

Since the yield function is an isotropic function of its arguments, the consistency condition can be written as [9]

\[
\dot{\Phi} = \frac{\partial \Phi}{\partial \sigma} \dot{\sigma} + \frac{\partial \Phi}{\partial \varepsilon^p} \dot{\varepsilon} + \frac{\partial \Phi}{\partial f} \dot{f} + \sum_{i=1}^{2} \frac{\partial \Phi}{\partial w_z} \dot{w}_z + \sum_{i=1}^{3} \frac{\partial \Phi}{\partial \mathbf{n}^{(i)}} \dot{\mathbf{n}}^{(i)}
\]

\[
= \frac{\partial \Phi}{\partial \sigma} \dot{\sigma} + \frac{\partial \Phi}{\partial \varepsilon^p} \dot{\varepsilon} + \frac{\partial \Phi}{\partial f} \dot{f} + \sum_{i=1}^{2} \frac{\partial \Phi}{\partial w_z} \dot{w}_z + \sum_{i=1}^{3} \frac{\partial \Phi}{\partial \mathbf{n}^{(i)}} \dot{\mathbf{n}}^{(i)}
\]

\[
= \frac{\partial \Phi}{\partial \sigma} \dot{\sigma} + \frac{\partial \Phi}{\partial \varepsilon^p} \dot{\varepsilon} + \frac{\partial \Phi}{\partial f} \dot{f} + \sum_{i=1}^{2} \frac{\partial \Phi}{\partial w_z} \dot{w}_z = 0,
\]

where \( \mathbf{n}^{(i)} = 0 \).

In the following we discuss the evaluation of \( \partial \Phi/\partial f \), \( \partial \Phi/\partial w_z \) and \( \partial \Phi/\partial \mathbf{n}^{(i)} \). Using the expression for \( \mathbf{m} \) given in Eq. (C.2) we conclude readily that

\[
\frac{\partial \Phi}{\partial f} = \frac{3}{(1 - f)^2} \sigma : \left( \frac{1}{2} \mathbf{K} + \frac{1 + f}{1 - f} \mu \mathbf{Q}^{-1}|_{r=1/2} \right) : \sigma.
\]

Starting with the identity \( \mathbf{Q} : \mathbf{Q}^{-1} = \mathbf{I} \) we conclude that

\[
\frac{\partial \left( \mathbf{Q}^{-1} \right)}{\partial w_z} = -\mathbf{Q}^{-1} : \frac{\partial \mathbf{Q}}{\partial w_z} : \mathbf{Q}^{-1}, \quad a = 1, 2.
\]

Therefore

\[
\frac{\partial \Phi}{\partial w_z} = -\frac{3f \mu}{(1 - f)^2} \sigma : \left( \mathbf{Q}^{-1} : \frac{\partial \mathbf{Q}}{\partial w_z} : \mathbf{Q}^{-1} \right)|_{r=1/2}, \quad a = 1, 2.
\]
Taking into account the definition of $Q$ given in Eq. (A.10) and the fact that
\[
\frac{\partial |Z^{-1} \cdot \xi|}{\partial w_z} = - \frac{(\xi \cdot n^{(z)})^2}{w_z^3 |Z^{-1} \cdot \xi|}, \quad \alpha = 1, 2,
\]
we reach after some algebra the following formula:
\[
\frac{\partial Q}{\partial w_x} = - \frac{Q}{w_z} + \frac{3c}{w_z^2} \int_{|\xi|=1} E(\xi) \frac{(\xi \cdot n^{(z)})^2}{|Z^{-1} \cdot \xi|^3} dS(\xi), \quad \alpha = 1, 2,
\]
where $c = 1/(4\pi w_1 w_2)$. Summarizing, we mention that $\partial \Phi/\partial w_x$ is determined from Eq. (C.6) together with (C.8).

Similarly, it can be shown readily that the derivatives $\partial \Phi/\partial n^{(j)}$ are determined from the relations
\[
\frac{\partial \Phi}{\partial n^{(j)}} = - \frac{3f \mu}{(1-f)^2} \sigma : \left( Q^{-1} : \frac{\partial Q}{\partial n^{(j)}} : Q^{-1} \right)_{\sigma = 1/2}, \quad i, j = 1, 2, 3
\]
and
\[
\frac{\partial Q}{\partial n^{(j)}} = \frac{3c}{w_i^2} \int_{|\xi|=1} E(\xi) \frac{\xi_j (\xi \cdot n^{(z)})}{|Z^{-1} \cdot \xi|^3} dS(\xi) \quad \text{(no sum over $i$)}, \quad i, j = 1, 2, 3,
\]
where $c = 1/(4\pi w_1 w_2)$ and $w_3 = 1$.

The matrix mapping of tensorial expressions such as those used above are discussed in detail by Nadeau and Ferrari [38].

References


